A NEW FOUNDATION OF THE PROJECTIVE DIFFERENTIAL THEORY OF CURVES IN FIVE-DIMENSIONAL SPACE

BY

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Introduction. The purpose of the present paper is to develop a purely geometric theory of the projective differential geometry of curves in a space of five dimensions, the methods hitherto adopted by various authors(2) being more or less analytic and artificial.

The projective differential theory of plane curves is one of the cornerstones upon which our theory of curves in five dimensions rests. The problem of finding a covariant figure to associate projectively with the neighborhood of order five at an ordinary point of a plane curve is an interesting one as has been pointed out by Levi-Civita and Fubini [9]. We have succeeded in solving this problem in an elementary way. Then using neighborhoods of order *six* we have obtained, besides a covariant triangle, three covariant points I_i $(i=1, 2, 3)$ any one of which can be selected as unit point.

In five-dimensional space the osculating plane p at an ordinary point P of a curve Γ intersects the developable hypersurface of Γ in a plane curve C, of which P is either an ordinary point or a k-ic $(k=6, 7, 8)$ point [2]. In any case a covariant unit point I and a covariant triangle $\{PP_1P_2\}$ can be determined in p, PP_1 being the tangent to Γ at P.

When a plane p, passing through PP_1 and lying in the osculating three-space of Γ at P, rotates about PP_1, the Bompiani osculant O_6 [1], associated with the point of inflexion P of the projection produced by projecting Γ on the plane p from the point $\alpha P+\beta P_2$, constitutes a generator of a covariant quadric Q_3. Since P, P_1, P_2 are three vertices of a quadrilateral on Q_3, we may take the fourth vertex for P_3. In a similar way we define two other vertices P_4 and P_5 of a covariant pyramid $\{PP_1P_2P_3P_4P_5\}$ and two other covariant quadrics Q_4 and Q_5 which pass through the quadrilaterals $PP_1P_3P_4P$ and $PP_1P_4P_5P$ respectively.

Let I be the unit point in the plane PP_1P_2; then the unit point in the space may be defined by the fact that the quadrics Q_k $(k=3, 4, 5)$ pass through $P+P_1+P_2+P_k$ respectively.

This accomplished, the projective Frenet-Serret formulae follow immediately from the canonical expansions of Γ. Our method has the advantage

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(2) Wilczynski [15], Sannia [10], Su [11, 12]. Numbers in brackets refer to the Bibliography at the end of the paper.

that the geometric significance of the projective invariants of the curve can easily be brought to light.

CHAPTER I. THE PROJECTIVE DIFFERENTIAL THEORY OF PLANE CURVES

In this chapter we shall discuss some results useful for the study of curves in a five-dimensional projective space. In §1 a covariant triangle of reference and a covariant unit point are constructed by means of the neighborhood of order *six* at a point of an ordinary plane curve. This leads us in §2 to reconstruct Lane's [8] canonical expansion of a plane curve at its sextactic point.

1. A new projective differential theory of plane curves. In this section we study plane curves in the neighborhood of an ordinary point, obtaining a canonical expansion, the Frenet-Serret formulae and an interpretation of the invariants.

1.1. *The covariant triangle of reference.* Let x and y be nonhomogeneous coordinates of a point in a plane. If $x=x(u)$, $y=y(u)$ are the parametric equations of a plane curve C, the expansion of C at an ordinary point P of C with respect to a local triangle of reference $\{PP_1P_2\}$ is

$$y = ax^2 + bx^3 + cx^4 + dx^5 + ex^6 + fx^7 + (8), \tag{1}$$

where PP_1 is the tangent to C at P. We take P_2 on the osculating conic C_2 of C at P and P_1P_2 as the tangent to the conic C_2 at P_2, so that (1) reduces to

$$y = ax^2 + dx^5 + ex^6 + fx^7 + (8). \tag{2}$$

It is well known that there exists a bundle of seven-point cubics of C at P:

$$\alpha y(y - ax^2) + \beta(y - ax^2 - dxy^2/a^2 - ey^3/a^3) + \gamma(ax^3 - xy + dy^3/a^3) = 0, \tag{3}$$

where α, β, γ are arbitrary constants. $y=0$ and $y-ax^2=0$ respectively intersect (3), besides P, at a pair of related points

$$(\beta/\gamma, 0) \tag{4}$$

and

$$(\beta d/(\gamma d - \beta c), a\beta^2 d^2/(\gamma d - \beta e)^2), \tag{5}$$

whose join passes through a fixed point

$$(d/e, ad^2/e^2). \tag{6}$$

For $d\neq 0$, we select (6) for P_2 and reduce (2) to

$$y = ax^2 + dx^5 + fx^7 + (8).$$

1.2. *Covariant figure associated with an ordinary point of a plane curve.* Let x^1, x^2, x^3 be the homogeneous coordinates of a point defined by

$$x = x^1/x^3, \qquad y = x^2/x^3.$$

The polar line of $A(\alpha, 0, 1)$ with respect to C_2,

$$x^2 = 2a\alpha x^1, \tag{7}$$

intersects C_2 at (0, 0, 1) and

$$A'(1, 2a\alpha, 1/2\alpha) \tag{8}$$

respectively. At P a six-point cubic of C can be determined to be tangent to the lines (7) and AA',

$$-2a\alpha x^1 + x^2/2 + 2a\alpha^2 x^3 = 0, \tag{9}$$

at (8) and $(\alpha, 0, 1)$ respectively; its equation is found to be

$$(y - 2a\alpha x)^2(-2a\alpha x + y/2 + 2a\alpha^2) + (y - 2a\alpha x)(-4d\alpha^3 y^2/a) - 2y(-2a\alpha x + y/2 + 2a\alpha^2)^2 = 0. \tag{10}$$

As α varies the envelope of the cubic (10) *is a covariant figure*:

$$4a^5(ax^2 - y)^3 + d^2y^6 = 0. \tag{11}$$

1.3. *The canonical expansion*. The tangent of the conic $y - ax^2 = 0$ at the covariant point (6) intersects $y = 0$ at $(d/2e, 0, 1)$. A triangle of vertices P, (6), and $(d/2e, 0, 1)$ is called a fundamental triangle. We denote (6) and $(d/2e, 0, 1)$ by P_2 and P_1 respectively. If the fundamental triangle is taken for reference, the expansion of the curve C at P then becomes

$$y = ax^2 + dx^5 + fx^7 + (8).$$

As we have given in §1.2, the two consecutive points (1, 0, 0) and $(\alpha, 0, 1)$ $(\alpha \to \infty)$ on PP_1 correspond to the cubics

$$-dxy^2 - a^3x^2 + a^2y = 0 \tag{12}$$

and (10) respectively. They intersect at the points on the cubic

$$a^3x^3 - a^2xy + dy^3/2a = 0, \tag{13}$$

namely, P and the points of coordinates

$$(\omega^k(d/2a^4)^{1/3}, \quad 0, \quad 2a\omega^{2k}(d/2a^4)^{2/3}) \qquad (k = 1, 2, 3, \quad \omega^3 = 1, \omega \neq 1). \tag{14}$$

If one of the latter is taken for unit point I, then we have

$$d = 1/8, \qquad a = 1/2.$$

The coordinate system thus obtained will be called the canonical coordinate system, and the coordinates of a point referred to it the canonical coordinates.

The canonical expansion of C at P now becomes

$$y = x^2/2 + x^5/8 + k(u)x^7 + (8). \tag{15}$$

1.4. *Projective Frenet-Serret formulae*. Let $(x^1(u), x^2(u), x^3(u))$, $(x_1^1(u),$

$x_1^2(u)$, $x_1^3(u)$), and ($x_2^1(u)$, $x_2^2(u)$, $x_2^3(u)$) be the coordinates of the vertices P, P_1, P_2 of the fundamental triangle. In order to preserve $x^j(u)+x_1^j(u)+x_2^j(u)$ $(j=1, 2, 3)$ as the selected unit point associated with the canonical triangle of reference $\{P(u), P_1(u), P_2(u)\}$, we are unable to multiply the coordinates of the three points $P(u)$, $P_1(u)$ and $P_2(u)$ by different proportionality factors. On the contrary, there exists a unique function $\lambda(u)$ such that

$$d(\lambda(u)x^j(u))/du = \lambda(u)\alpha(u)x_1^j(u) \qquad (j = 1, 2, 3). \tag{16}$$

For brevity we still use $x^j(u)$, $x_1^j(u)$, $x_2^j(u)$ instead of $\lambda(u)x^j(u)$, $\lambda(u)x_1^j(u)$, $\lambda(u)x_2^j(u)$, so that (16) becomes

$$dx^j(u)/du = \alpha(u)x_1^j(u). \tag{16'}$$

Through the independence of the three points P, P_1, P_2, $dx_1^j(u)/du$ and $dx_2^j(u)/du$ may be expressed in the form

$$\begin{aligned} \frac{dx_1^j(u)}{du} &= \beta x^j(u) + \gamma x_1^j(u) + \delta x_2^j(u), \\ \frac{dx_2^j(u)}{du} &= \mu x^j(u) + \nu x_1^j(u) + \tau x_2^j(u). \end{aligned} \tag{16''}$$

But the condition of immovability and (15) show that

$$\begin{aligned} \beta &= 112k(u)\alpha(u), & \gamma &= 0, \\ \delta &= \alpha(u), & \mu &= 5\alpha(u)/2, \\ \nu &= 112k(u)\alpha(u), & \tau &= 0. \end{aligned}$$

Hence

$$\begin{aligned} \frac{dx}{du} = \alpha(u)x_1, \qquad \frac{dx_1}{du} &= \alpha(u)(112k(u)x + x_2), \\ \frac{dx_2}{du} &= \alpha(u)\left(\frac{5}{2}x + 112k(u)x_1\right), \end{aligned} \tag{17}$$

where we have written x and x_i $(i=1, 2)$ in place of x^j and x_i^j $(j=1, 2, 3; i=1, 2)$.

With respect to a parametric transformation $\bar{u}=f(u)$, $\alpha(u)du$ and $k(u)$ are clearly intrinsic.

A proper projective transformation

$$x^j = \sum_{k=1}^{3} a_k^j y^k \qquad (j = 1, 2, 3), \tag{18}$$

where $|a_k^j| \neq 0$, carries $x^j(u)$, $x_1^j(u)$ and $x_2^j(u)$ $(j=1, 2, 3)$ into the points $y^j(u)$, $y_1^j(u)$ and $y_2^j(u)$ respectively. From the first two equations of (17), we find

$$\frac{d}{du}\left(\sum_{k=1}^{3} a_k^j y^k\right) = \alpha(u)\left(\sum_{k=1}^{3} a_k^j y_1^k\right),$$

$$\frac{d}{du}\left(\sum_{k=1}^{3} a_k^j y^k\right) = \alpha(u)\left(112k(u)\sum_{k=1}^{3} a_k^j y^k + \sum_{k=1}^{3} a_k^j y^k\right),$$

or, because of $|a_j^i| \neq 0$,

$$\frac{dy}{du} = \alpha y_1, \qquad \frac{dy_1}{du} = \alpha(112k(u)y + y_2).$$

That is, αdu and $k(u)$ are intrinsic and invariant. Consequently, we call them the projective arc-element and the projective curvature respectively.

Since any one of the points

$$\left(\omega^k\left(\frac{d}{2a^4}\right)^{1/3},\ 1,\ \frac{\omega^{2k}}{a^2}\left(\frac{d}{2a^4}\right)^{2/3}\right) \qquad (k = 1, 2, 3) \tag{19}$$

may be selected as the unit point, there are evidently three kinds of canonical coordinates. In consequence, the vertices of the fundamental triangle may have different canonical coordinates $z^j(u)$, $z_1^j(u)$, $z_2^j(u)$ and $x^j(u)$, $x_1^j(u)$, $x_2^j(u)$ $(j=1, 2, 3)$ respectively. They are, however, connected by

$$x^j = z^j/\omega^2, \qquad x_1^j = z_1^j/\omega, \qquad x_2^j = z_2^j \qquad (j = 1, 2, 3),$$

where $\omega^3=1, \omega\neq 1$.

The equations (17) then become

$$\frac{dz}{du} = \alpha\omega z_1, \qquad \frac{dz_1}{du} = \alpha\omega\left(\frac{112k(u)}{\omega^2}z + z_2\right),$$

$$\frac{dz_2}{du} = \alpha\omega\left(\frac{5}{2}z + 112k(u)\cdot\frac{1}{\omega^2}z_1\right).$$

Therefore the projective arc-element and the projective curvature are multiplied by ω if another point among (19) is taken as the unit point.

If we put $d\sigma=\alpha du$, (17) reduces to the projective Frenet-Serret formulae required:

$$\frac{dx}{d\sigma} = x_1, \qquad \frac{dx_1}{d\sigma} = 112k(\sigma)x + x_2, \qquad \frac{dx_2}{d\sigma} = \frac{5}{2}x + 112k(\sigma)x_1. \tag{A}$$

Eliminating x_1 and x_2 from (A) gives

$$-\frac{d^3x}{d\sigma^3} + 224k(\sigma)\frac{dx}{d\sigma} + \left(112\ \frac{dk(\sigma)}{d\sigma} + \frac{5}{2}\right)x = 0, \tag{B}$$

which is preserved under the transformation

$$\bar{x} = \omega^2 x, \qquad \bar{k}(\sigma) = \omega k(\sigma), \qquad d\bar{\sigma} = \omega d\sigma.$$

Thus an analytic function $k(\sigma)(a \leqq \sigma \leqq b)$ corresponds except for a projectivity to an analytic curve, such that $d\sigma$ is the projective arc-element, $k(\sigma)$ the projective curvature, and independent solutions of the differential equation (B) the canonical coordinates of a current point of the curve. Conversely, an analytic curve corresponds to a unique projective curvature and a unique projective arc-element.

1.5. *Geometrical interpretation of the projective curvature and the projective arc-element.* By use of the unit point I we can obtain the following interpretations(3) of the invariants appearing in (A):

$$x_1(x, x_2; dx_1/d\sigma, I) = 112k, \qquad x_2(x, x_1; x + dx, I) = d\sigma + (2),$$

where the left members are cross ratios of the indicated pencils and (2) denotes an infinitesimal of higher order.

2. **A generalization of sextactic points.** In a recent paper [2] we have represented the neighborhood of various orders of a plane curve C at its sextactic point P. A cusped quartic Q_4 is utilized such that its cusp lies on the tangent of C at P and its cuspidal tangent is also tangent to the hyperosculating conic of C at P. Since either the sextactic point or its generalization appear in the theory of curves in five-dimensional space, we find it convenient to give here an outline of the theory on the generalized sextactic points.

2.1. *The canonical expansion of a plane curve at its sextactic point.* At a sextactic point $P(0, 0, 1)$ of a plane curve C the osculating conic has a contact with the curve of order five. Hereafter we select the triangle of reference $\{PP_1P_2\}$ in such a way that this osculating conic is tangent to PP_1 and P_1P_2 at P and P_2 respectively, so that the expansion of Γ becomes

$$y = ax^2 + ex^6 + fx^7 + gx^8 + hx^9 + (10), \tag{20}$$

and the equation of the osculating conic is $y - ax^2 = 0$.

The quartic Q_4, which has at P a contact of order six with (20), is easily found to be

$$\begin{aligned} Dy^4 + Ey^3(\alpha y - 2ax) - \frac{8e}{a\alpha} y^3\left(2ax - \frac{2a}{\alpha} - \frac{\alpha y}{2}\right) \\ + \left(2ax - \frac{2a}{\alpha} - \frac{\alpha y}{2}\right)^2 (4a^2x^2 - 4ay) = 0, \end{aligned} \tag{21}$$

provided that Q_4 has a cusp at $A(1, 0, \alpha)$. The polar line l of A with respect to the conic $y - ax^2 = 0$ intersects it again at $B(2/\alpha, 4a/\alpha^2)$. If Q_4 passes through B, the locus of the common points between Q_4 and l consists of the conic and a covariant quartic

(3) For other interpretations see F. Vychyclo [13, 14].

$$-ey^4/a^4+(ax^2-y)^2=0. \tag{22}$$

The latter is particularly useful in determining the unit point.

When Q_4 given by (21) has at P a contact of order seven with (20), then

$$E=(8e/a\alpha-8f/a\alpha^2).$$

Furthermore, we can show that the join of the two common points, beside P, of Q_4 and the osculating conic passes through a fixed point $A'[-e, 0, -2ae+f]$. The correspondence between A and A' is a projectivity with unique double point $H(e, 0, f)$, namely, the Halphen point discovered by Lane [8]. When it is taken for (1, 0, 0), we have $f=0$. We select further the unit point on the osculating conic and (1, 1, 0) on (22), so that

$$a=1, \qquad e=1, \tag{23}$$

and the canonical expansion of the plane curve at its sextactic point becomes

$$y=x^2+x^6+(8). \tag{24}$$

It should be noted that the selection of the unit point so as to satisfy (23) is not unique, but one of the three points $(-1, 1)$, $(i, -1)$ and $(-i, -1)$ can be taken for the unit point without changing the form (24).

2.2. *The canonical expansion of a plane curve at a generalized sextactic point.* A point P on a curve C is called a k-ic point if the osculating conic has at P a k-point conic with the curve C ($k\geqq 6$). Neither Lane's method nor ours in the foregoing section can be utilized in this case. We shall now determine the canonical expansions of a plane curve at a 7-ic or 8-ic point.

(i) *The 7-ic point.* A necessary and sufficient condition that P is a 7-ic point is $af\neq 0$ and $e=0$. If the quartic (21) has at P a contact of order eight with (20), the equation of the quartic (21) takes the form

$$\begin{aligned}\left(-\frac{6f\alpha}{a}+\frac{4g}{a}\right)y^4-\frac{8f}{a}y^3(\alpha y-2ax)\\ +4a(ax^2-y)\left(2a\alpha x-2a-\frac{\alpha^2 y}{2}\right)^2=0.\end{aligned} \tag{21'}$$

The polar line of A with respect to the osculating conic of C at P intersects (21′) at three points, and when α varies, one of them describes the curve

$$\left(\frac{-12f}{a^2}xy+\frac{4g}{a^3}y^2\right)y^4+4a(ax^2-y)^3=0. \tag{25}$$

This curve intersects the osculating conic in two different points on the line

$$-3afx+gy=0. \tag{26}$$

Evidently, g reduces to zero if (26) is taken for the side $x=0$ of the triangle of reference.

In order that the unit point be on the osculating conic and (1, 1, 0) on the curve (25), it is necessary and sufficient that

$$a = 1, \qquad f = -2/3. \tag{27}$$

The canonical expansion of the curve C then becomes

$$y = x^2 - 2x^7/3 + (9). \tag{28}$$

It is worth noticing that any one of the four points

$$((1/2)^{1/5}e^{2k\pi i/5},\ (1/4)^{1/5}e^{4k\pi i/5}) \qquad (k = 1, 2, 3, 4)$$

may be used for the unit point without changing the form (28).

(ii) *The 8-ic point.* If P is an 8-ic point, then

$$ag \neq 0, \qquad e = 0, \qquad f = 0.$$

Neither the Halphen point nor the covariant line (26) can be determined. We require a further modification in the derivation of the canonical expansion.

In this case (21′) becomes

$$4gy^4/a + 4a(ax^2 - y)(2a\alpha x - 2a - \alpha^2 y/2)^2 = 0, \tag{21''}$$

which has at P a contact of order nine with C if

$$\alpha = h/2g. \tag{29}$$

Thus we arrive at a covariant point $(2g, 0, h)$. Upon selecting S for (1, 0, 0), α and h reduce to zero and the corresponding 10-point quartic of S is found to be

$$gy^4/a + a^3(ax^2 - y) = 0. \tag{30}$$

If the common points of $x=0$ and (30) are projected from S to the osculating conic, we obtain six points of coordinates

$$(\pm (1/a)^{1/2}(g/a^4)^{1/6}\omega^{k/2},\ 1,\ (g/a^4)^{1/3}\omega^k) \qquad (k = 1, 2, 3,\ \omega^3 = 1,\ \omega \neq 1). \tag{31}$$

By choosing any one of them for the unit point both a and g reduce to 1, so that the canonical expansion of C in the neighborhood of P is found to be

$$y = x^2 + x^8 + (10). \tag{32}$$

Chapter II. The projective differential theory of curves in five-dimensional space

In §3 we construct a covariant pyramid of reference at an ordinary point of a curve in five-dimensional space. In addition to some simple interpretations of the invariants of the curve the canonical expansions of five different types are given.

3. **The covariant pyramid.** Suppose that the point $P=P(u)$ describes a curve Γ in five-dimensional space where u is a parameter. Take $\{PP_1P_2P_3P_4P_5\}$ for the local pyramid of reference associated with a generic point P of Γ, such that P_j lies in the osculating linear space $S_j(P)$ of j dimensions at P to Γ, but not in the osculating space $S_{j-1}(P)$. Any point Q in the space can be expressed by $\sum_{\tau=0}^{5} X_\tau P_\tau$, where $P_0 \equiv P$.

The nonhomogeneous coordinates are defined by putting

$$x_1 = X_1/X_0,\ x_2 = X_2/X_0,\ \cdots,\ x_5 = X_5/X_0.$$

In case the parametric equation of the curve is of the form

$$X(u) = \sum_{\tau=0}^{5} X_\tau(u) P_\tau,$$

the expansions of the nonhomogeneous coordinates of Γ are

$$x_i = \sum_{j=0}^{\infty} b_{i,i+j}(x_1)^{i+j} \qquad \left(\prod_{2}^{5} b_{i,i} \neq 0,\ i = 2, 3, 4, 5\right), \tag{1}$$

where $b_{i,i+j}$ are local functions.

3.1. *The developable hypersurface of* Γ. Since x_1 depends upon u, we may regard x_1, λ_1, λ_2 and λ_3 as parameters and obtain the parametric equations of the developable hypersurface of Γ:

$$\begin{aligned} X_0 &= 1, \qquad X_1 = x_1 + \lambda_1, \\ X_k &= x_k + \sum_{\nu=1}^{3} \lambda_\nu \frac{d^\nu x^k}{(dx_1)^\nu} \qquad (k = 2, 3, 4, 5). \end{aligned} \tag{2}$$

The section C of this hypersurface produced by the osculating plane of Γ at its ordinary point P is given by the equations

$$\begin{aligned} X_0 &= D[dx_3/dx_1,\ dx_4/dx_1,\ dx_5/dx_1], \qquad X_1 = D[x_1, x_3, x_4, x_5], \\ X_2 &= D[x_2, x_3, x_4, x_5], \qquad X_3 = 0, \qquad X_4 = 0, \qquad X_5 = 0, \end{aligned} \tag{3}$$

where $D[\cdots]$ denotes the Wronskian.

Putting, for brevity,

$$\xi_0 = \frac{X_0}{x_1^6} = \sum_0^\infty B_\nu x_1^\nu, \quad \xi_1 = \frac{X_1}{x_1^6} = x_1 \sum_0^\infty C_\nu x_1^\nu, \quad \xi_2 = \frac{X_2}{x_1^6} = x_1^2 \sum_0^\infty D_\nu x_1^\nu, \tag{4}$$

we find that if P_2 lies on the osculating conic C_2 of C and if P_1P_2 is the tangent to C_2, then

$$\begin{aligned} 2D_0B_0C_1 - C_0D_0B_1 - C_0D_1B_0 &= 0, \\ D_0B_0(C_1^2 + 2C_0C_2) - C_0^2B_2D_0 - D_1B_1C_0^2 - B_0D_2C_0^2 &= 0. \end{aligned} \tag{5}$$

Consequently, the osculating conic of Γ at P is given by the equations

$$(6)\qquad B_0D_0X_1^2/C_0^2 - X_0X_2 = 0, \qquad X_3 = 0, \qquad X_4 = 0, \qquad X_5 = 0.$$

A simple calculation gives

$$B_0 = 120 \prod_3^5 b_{i,i}, \qquad C_0 = 48 \prod_3^5 b_{i,i},$$

$$D_0 = 12 \prod_2^5 b_{i,i}, \qquad B_1 = 2\cdot 6^3\cdot\left(\prod_3^5 b_{i,i}\right)\cdot\frac{b_{5,6}}{b_{5,5}},$$

$$(7)\qquad C_1 = 180\left(\prod_3^5 b_{i,i}\right)\frac{b_{5,6}}{b_{5,5}}, \qquad D_1 = 48\left(\prod_2^5 b_{i,i}\right)\cdot\frac{b_{5,6}}{b_{5,5}},$$

$$B_2 = -3^3\cdot 20 b_{3,3}b_{4,6}b_{5,5} + 7\cdot 144 b_{3,3}b_{4,4}b_{5,7},$$

$$C_2 = -240 b_{3,3}b_{4,6}b_{5,5} + 3^3\cdot 2^4 b_{3,3}b_{4,4}b_{5,7},$$

$$D_2 = -72 b_{2,2}b_{3,3}b_{4,6}b_{5,5} + 120 b_{2,2}b_{3,3}b_{4,4}b_{5,7}.$$

Substituting them in (5) and (6), we have

$$(8)\qquad b_{5,6} = 0,$$

and

$$(9)\qquad 5b_{5,5}b_{4,6} - 4b_{4,4}b_{5,7} = 0.$$

When P is an ordinary point of C, we define the fundamental triangle $\{PP_1P_2\}$ by the method stated in §1.1. The cubic,

$$(10)\qquad \left(\frac{D_0B_0}{C_0^2}X_1^2 - X_0X_2\right)X_0 - X_1X_2^2\left(\frac{2C_3B_0^2}{C_0^2D_0} - \frac{B_0^2D_3}{C_0D_0^2} - \frac{B_0B_3}{C_0D_0}\right) = 0,$$

which is determined by the conditions: (i) it passes through P_1; (ii) it is tangent to (6) at P_2; and (iii) it has at P a contact of order five with C, should have at P a contact of order six with C. In other words,

$$(11)\qquad (D_0B_0/C_0^2)(C_2^2 + 2C_0C_4) - D_0B_4 - D_4B_0 - D_2B_2 = 0.$$

Otherwise, for $A \neq 0$,

$$(12)\qquad \left(\frac{D_0B_0}{C_0^2}X_1^2 - X_0X_2\right)X_1 - X_1X_2^2\left(\frac{2C_3B_0^2}{C_0^2D_0} - \frac{B_0^2D_3}{C_0D_0^2} - \frac{B_0B_3}{C_0D_0}\right) + AX_2^3 = 0$$

would have at P a contact of order six with C, which contradicts the hypothesis that P_1P_2 does not pass through any point common to (12) and the osculating conic.

3.2. *The construction of the quadric Q_3.* A plane passing through PP_1 and lying in the osculating three-space of Γ at P is given by the equations

$$(13)\qquad X_3 - \mu X_2 = 0, \qquad X_4 = 0, \qquad X_5 = 0.$$

If the osculating planes of Γ be projected from a point $(1, 0, \rho, 0, 0, 0)$ on the above plane, we obtain a plane curve Γ_1, whose equations are easily found to be (13) and

$$(14)\qquad \begin{aligned} X_1 &= \frac{1}{x_1^6} D[x_1, x_4, x_5], \qquad X_3 = \frac{1}{x_1^6} D[x_3, x_4, x_5], \\ X_0 &= \frac{1}{x_1^6} D\left[\frac{dx_4}{dx_1}, \frac{dx_5}{dx_1}\right] + \frac{1}{\rho x_1^6} D\left[x_4, x_5, \frac{1}{\mu} x_3 - x_2\right]. \end{aligned}$$

Put

$$(15)\qquad \begin{aligned} X_1 &= x_1 \sum_0^\infty E_\nu x_1^\nu, \qquad X_3 = x_1^3 \sum_0^\infty F_\nu x_1^\nu, \\ X_0 &= \sum_0^\infty G_\nu x_1^\nu + \frac{1}{\rho\mu} x_1^3 \sum_0^\infty F_\nu x_1^\nu - \frac{x_1^2}{\rho} \sum_0^\infty H_\nu x_1^\nu. \end{aligned}$$

Then

$$\begin{aligned} E_\nu &= \sum_{\nu_1+\nu_2=\nu} b_{4,4+\nu_1} b_{5,5+\nu_2} (3+\nu_1)(4+\nu_2)(1+\nu_2-\nu_1), \\ F_\nu &= \sum_{\nu_1+\nu_2+\nu_3=\nu} b_{3,3+\nu_1} b_{4,4+\nu_2} b_{5,5+\nu_3} P(\nu_1, 1+\nu_2, 2+\nu_3), \\ G_\nu &= \sum_{\nu_1+\nu_2=\nu} (4+\nu_1)(5+\nu_2) b_{4,4+\nu_1} b_{5,5+\nu_2} (1+\nu_2-\nu_1), \\ H_\nu &= \sum_{\nu_1+\nu_2+\nu_3=\nu} - b_{2,2+\nu_1} b_{5,5+\nu_2} b_{4,4+\nu_3} P(\nu_1, 2+\nu_3, 3+\nu_2) \end{aligned}$$

where

$$P(\mu_1, \mu_2, \cdots, \mu_n) = \prod_{i>j\geqq 1}^{n} (\mu_i - \mu_j).$$

For the subsequent discussion it is convenient to give here an outline of the theory of singularities of a plane curve [5, 6]. If the expansions of the coordinates of a plane curve are

$$x = s\sum_0^\infty a_\nu s^\nu, \qquad y = s^m \sum_0^\infty b_\nu s^\nu, \qquad z = 1 + \sum_1^\infty c_\nu s^\nu,$$

the point $(0, 0, 1)$ is a singular point of the curve denoted by S_1^m. When $(m-3)$ conditions are satisfied, this singularity is particularly simple. Let $U_{10,k}$ and $A_{m,k}$ be respectively the coefficients of s^k in the expansions of $(\sum_0^\infty b_\nu s^\nu) \cdot (1+\sum_1^\infty c_\nu s^\nu)^{m-1}$ and $(\sum_0^\infty a_\nu s^\nu)^m$. If

$$A_{m,1} - a_0^m U_{10,1}/b_0 = 0,$$

the coordinates of the covariant points O_{m+1} and O_{2m} of the curve are $(1, 0, 0)$ and

$$(16)\quad \left(\frac{1}{ma_0^{m-1}b_0}\left(A_{m,m-1}-\frac{\overset{m}{a_0}}{b_0}U_{10,m-1}\right),1,\frac{1}{a_0^m b_0(m-1)}\left(\frac{\overset{m}{a_0}}{b_0}U_{10,m}-A_{m,m}\right)\right).$$

Moreover, the conditions of representability become

$$(17)\qquad \overset{m}{a_0}U_{10,k}/b_0-A_{m,k}=0 \qquad (k=2,\cdots,m-2).$$

From (8) it follows that the osculant O_4 associated with the point of inflexion P of Γ_1 always coincides with P_1. The coordinates of O_6 are

$$X_4=0,\qquad X_5=0,\qquad X_3=1,\qquad X_2=1/\mu,$$

$$X_1=-\frac{1}{3}\left\{\frac{E_0F_2}{F_0^2}+\frac{2E_0G_2}{F_0G_0}-3\frac{F_2}{F_0}-2\frac{E_0H_0}{\rho F_0G_0}\right\},$$

$$X_0=\frac{\overset{2}{G_0}}{2E_0F_0}\left\{\frac{E_0}{F_0}\left(\frac{F_3}{G_0}+\frac{2F_0G_3}{G_0^2}-\frac{2F_0H_1}{\rho G_0^2}+2\frac{E_0F_0}{G_0^2\rho\mu}\right)-3\frac{E_3}{G_0}\right\}.$$

Eliminating μ we obtain[4] a generator of the quadric Q_3:

$$(18)\quad \begin{aligned}\frac{X_0}{X_3}&=\frac{1}{2}\left(\frac{G_0F_3}{F_0^2}+2\frac{G_3}{F_0}-3\frac{G_0E_3}{E_0F_0}\right)\\&\quad+\frac{F_0G_0}{2E_0H_0}\left(\frac{X_2}{X_3}-\frac{H_1}{F_1}\right)\left(\frac{3X_1}{X_3}+\frac{E_0F_2}{F_0^2}+2\frac{E_0G_2}{F_0G_0}-3\frac{E_2}{F_0}\right),\\&\qquad X_4=0,\qquad X_5=0.\end{aligned}$$

Since P, P_1 and P_2 lie on the same quadrilateral of the quadric, we take the fourth vertex as P_3. Then

$$(19)\quad \begin{aligned}G_0E_0F_3+2E_0F_0G_3-3G_0E_3F_0=0,\qquad H_1=0,\\ E_0F_2G_0+2E_0F_0G_2-3G_0F_0E_2=0.\end{aligned}$$

It is easily verified by means of (8) that

$$E_0=12b_{4,4}b_{5,5},\qquad F_0=2b_{3,3}b_{4,4}b_{5,5},$$
$$G_0=20b_{4,4}b_{5,5},\qquad H_1=3b_{2,3},\qquad E_2=-20b_{4,6}b_{5,5}+54b_{4,4}b_{5,7},$$
$$F_2=-6b_{3,3}b_{4,6}b_{5,5}+12b_{3,3}b_{4,4}b_{5,7},\qquad G_2=-30b_{5,5}b_{4,6}+84b_{4,4}b_{5,7}.$$

Hence the last two equations of (19) become

$$(20)\qquad b_{2,3}=0,$$

$$(21)\qquad -310b_{5,5}b_{4,6}+729b_{4,4}b_{5,7}=0.$$

Combining (19) and (21) we find

(4) This is a result analogous to what we obtained in 1942 in studying the point of inflexion of a space curve, cf. Chang [3].

(22) $$b_{4,6} = 0, \qquad b_{5,7} = 0$$

and consequently

$$F_3 = 2b_{3,6}b_{4,4}b_{5,5} - 16b_{3,3}b_{4,7}b_{5,5} + 20b_{3,3}b_{4,4}b_{5,8},$$
$$G_3 = -70b_{5,5}b_{4,7} + 128b_{4,4}b_{5,8}, \qquad E_3 = -48b_{5,5}b_{4,7} + 48b_{4,4}b_{5,8}.$$

Therefore the first equation of (19) becomes

(23) $$b_{3,6}/b_{3,3} - 3b_{4,7}/b_{4,4} + 9b_{5,8}/5b_{5,5} = 0.$$

The covariant quadric (18) is now given by the equations

(24) $$X_4 = 0, \qquad X_5 = 0, \qquad X_0X_3 = (5b_{3,3}/6b_{2,2})X_1X_2.$$

3.3. *The construction of the quadric* Q_4. In a similar way we proceed to determine the vertex P_4 of the pyramid $\{P, P_1, \cdots, P_5\}$. In fact, let the osculating planes of Γ be projected from $(1, 0, 0, \rho, 0, 0)$ to a plane passing through PP_1 and lying in the osculating space of four dimensions of Γ at P:

(25) $$X_5 = 0, \qquad X_2 - \mu X_4 = 0, \qquad X_3 - \nu X_4 = 0.$$

The equations of the plane curve Γ_2 thus produced can be obtained by eliminating λ_1, λ_2 and λ_3 from (25) and

$$X_0 = 1 + \lambda_3, \qquad X_1 = x_1 + \lambda_1,$$
$$X_k = x_k + \lambda_1 dx_k/dx_1 + \lambda_2 d^2x_k/(dx_1)^2 \qquad (k = 2, 4, 5),$$
$$X_3 = x_3 + \lambda_1 dx_3/dx_1 + \lambda_2 d^2x_3/(dx_1)^2 + \lambda_3\rho.$$

The result of carrying out the computation shows that Γ_2 is given by (25) and

(26)
$$X_0 = \frac{1}{x_1^4}\left\{D\left[\frac{dx_5}{dx_1}, \frac{d(x_2 - \mu x_4)}{dx_1}\right] - \frac{1}{\rho}D[x_5, x_2 - \mu x_4, x_3 - \nu x_4]\right\},$$
$$X_1 = \frac{1}{x_1^4}\left\{x_1 D\left[\frac{dx_5}{dx_1}, \frac{d(x_2 - \mu x_4)}{dx_1}\right] - \frac{d}{dx_1}D[x_5, x_2 - \mu x_4]\right\},$$
$$X_4 = D[x_2, x_4, x_5].$$

For brevity, we put

$$D\left[\frac{dx_5}{dx_1}, \frac{dx_2}{dx_1}\right] = x_1^4 \sum_0^\infty E_\tau' x_1^\tau, \qquad D\left[\frac{dx_5}{dx_1}, \frac{dx_4}{dx_1}\right] = x_1^6 \sum_0^\infty F_\tau' x_1^\tau,$$
$$\frac{d}{dx_1}D[x_5, x_2] = x_1^5 \sum_0^\infty (6 + \tau)G_\tau' x_1^\tau, \qquad \frac{d}{dx_1}D[x_5, x_4] = x_1^7 \sum_0^\infty H_\tau' x_1^\tau,$$
$$D[x_5, x_3, x_2] = x_1^7 \sum_0^\infty I_\tau x_1^\tau,$$

so that (26) reduces to

$$
\begin{aligned}
X_0 &= \sum_0^\infty E'_\tau x_1^\tau - \mu x_1^2 \sum_0^\infty F'_\tau x_1^\tau - \frac{x_1^3}{\rho}\sum_0^\infty I_\tau x_1^\tau \\
&\quad + \frac{\nu x_1}{\rho}\sum_0^\infty H_\tau x_1^\tau - \frac{\mu}{\rho}\, x_1^5 \sum_0^\infty F_\nu x_1^\tau, \\
X_1 &= x_1 \sum_0^\infty E'_\tau x_1^\tau - x_1 \sum_0^\infty G'_\tau (\tau+6) x_1^\tau + \mu x_1^3 \sum_0^\infty [(\tau+8)H'_\tau - F'_\tau] x_1^\tau, \\
X_4 &= x_1^4 \sum_0^\infty H_\tau x_1^\tau.
\end{aligned} \tag{27}
$$

The point P is evidently a singularity [5] S_1^4 of Γ_2. Since the osculant Q_5 always coincides with P_1, the singular point S_1^4 of Γ_2 is representable when and only when

$$\frac{E'_0 - 6G'_0}{E'_0}\left(\frac{H_2}{H_0} + 3\,\frac{E'_2 - \mu F'_2}{E'_0}\right) - 4\,\frac{E'_2 - 8G'_2}{E'_0} - 4\mu\,\frac{8H'_0 - F'_0}{E'_0} = 0. \tag{28}$$

Therefore Γ_2 has a representable singular point at P if and only if the plane (25) belongs to the three-dimensional space $X_5 = 0$, $X_2 - \mu X_4 = 0$, where μ is given by (28). To select P_4 in this space brings μ to zero, and (28) to

$$\frac{E'_0 - 6G'_0}{E'_0}\left(\frac{H_2}{H_0} + 3\,\frac{E'_2 - \mu F'_2}{E'_0}\right) - 4\,\frac{E'_2 - 8G'_2}{E'_0} = 0. \tag{28'}$$

A simple calculation of Wronskians shows that

$$
\begin{aligned}
E'_0 &= -30 b_{5,5} b_{2,2}, & F'_0 &= -20 b_{5,5} b_{4,4}, & G'_0 &= -3 b_{2,2} b_{5,5}, \\
H'_0 &= -b_{5,5} b_{4,4}, & H_0 &= 6 b_{2,2} b_{4,4} b_{5,5}, & I_0 &= 6 b_{2,2} b_{3,3} b_{5,5}, \\
H_2 &= 0, & E'_2 &= -20 b_{4,4} b_{5,5}, & G'_2 &= -b_{2,4} b_{5,5}.
\end{aligned} \tag{29}
$$

Hence (28) becomes

$$b_{2,4} = 0. \tag{30}$$

The coordinates of the covariant point O_8 are easily found to be

$$
\begin{aligned}
&X_5 = 0, \qquad X_4 = 1, \qquad X_3 = \nu, \qquad X_2 = 0, \\
&X_1 = \frac{-E'_0}{4H_0}\left[\left(\frac{E'_0 - 6G'_0}{E'_0}\right)\left(\frac{H_3}{H_0} - 4\,\frac{E'_3}{E'_0}\right) + 3\,\frac{E'_3 - 9G'_3}{E'_0}\right] \\
&\qquad + \frac{3I_0}{4\rho H_0}\left(\frac{E'_0 - 6G'_0}{E'_0}\right), \\
&X_0 = \frac{1}{3}\left[\frac{E'_0 H_4}{H_0^2} + \frac{3E'_4}{H_0} - \frac{4E'_0}{H_0}\left(\frac{E'_4 - 10G'_4}{E'_0 - 6G'_0}\right)\right] + \frac{1}{\rho H_0}(-I_1 + \nu H_0),
\end{aligned} \tag{31}
$$

whose locus is a quadric Q_4:

$$\frac{X_0}{X_4}=\frac{1}{3}\left[-\frac{E_0' H_4}{H_0^2}+\frac{3E_4'}{H_0}-\frac{E_4'-10G_4'}{10H_0}\right]$$

$$(32)\qquad +\frac{4}{3}\left(\frac{E_0'}{E_0'-6G_0'}\right)\left(-\frac{I_1}{I_0}+\frac{H_0}{I_0}\frac{X_3}{X_4}\right)$$

$$\cdot\left[\frac{X_1}{X_4}+\frac{E_0'}{G_0'}\left(\frac{E_0'-6G_0'}{E_0'}\right)\left(\frac{H_3}{H_0}+\frac{3E_3'}{E_0'}\right)-4\,\frac{E_3'-6G_3'}{6H_0}\right],$$

$$X_5=0,\qquad X_2=0.$$

In order that $\{P_0P_1P_3P_4\}$ be a quadrilateral on Q_4, it is necessary and sufficient that

$$I_1=0,$$

$$((E_0'-6G_0')/E_0')(H_3/H_0+3\,(E_3'/E_0'))-4\,(E_3'-6G_3')/E_0'=0,$$

$$E_0'H_4/H_0^2+3E_4'/H_0-10(E_4'-10G_4')/H_0=0,$$

or

$$(33)\qquad b_{3,4}=0,$$

$$(34)\qquad 25b_{4,7}/b_{4,4}-18b_{5,8}/b_{5,5}=0,$$

$$(35)\qquad -20b_{2,6}/b_{2,2}+45b_{4,8}/b_{4,4}-28b_{5,9}/b_{5,5}=0,$$

because of (8), (20), (22), (23), (29), (34) and

$$E_3' = 96b_{2,2}b_{5,8},\qquad G_3' = -6b_{2,2}b_{5,8},$$
$$H_3 = (-30b_{4,7}/b_{4,4}+48b_{5,8}/b_{5,5})b_{2,2}b_{4,4}b_{5,5},$$
$$H_4 = (2b_{2,6}/b_{2,2}-54b_{4,8}/b_{4,4}+70b_{5,9}/b_{5,5})b_{2,2}b_{4,4}b_{5,5},$$
$$E_4' = (-126b_{5,9}/b_{5,5}+30b_{2,6}/b_{2,2})b_{2,2}b_{5,5},$$
$$G_4' = (-7b_{5,9}/b_{5,5}+b_{2,6}/b_{2,2})b_{2,2}b_{5,5}.$$

Putting

$$(36)\qquad b_{4,7}/b_{4,4}=72\omega_1,$$

we know by means of (23) and (34) that

$$(36')\qquad b_{5,8}/b_{5,5}=100\omega_1,\qquad b_{3,6}/b_{3,3}=36\omega_1.$$

3.4. *The construction of the quadric Q_5.* We come now to locate the remaining vertex P_5 of the covariant pyramid $\{P, P_1, P_2, P_3, P_4, P_5\}$. A plane p passing through PP_1 is, in general, given by the equations

$$(37)\qquad X_2-\nu_1X_5=0,\qquad X_3-\nu_2X_5=0,\qquad X_4-\nu_3X_5=0.$$

If a three-dimensional space contains the point $(1, 0, 0, 0, \rho, 0)$ and the osculating plane of Γ, then it must be given by the equations

$$X_0 = 1 + \lambda_1, \qquad X_1 = x_1 + \lambda_2, \qquad X_4 = x_4 + \sum_{\nu=1}^{2} \lambda_{\nu+1} \frac{d^\nu x_4}{(dx_1)^\nu} + \lambda_1 \rho,$$

(38)
$$X_k = x_k + \sum_{\nu=1}^{2} \lambda_{\nu+1} \frac{d^\nu x_k}{(dx_1)^\nu} \qquad (k = 2, 3, 5),$$

where x_1, λ_1, λ_2 and λ_3 are parameters. Eliminating λ_1, λ_2 and λ_3 from (37) and (38) we obtain a plane curve Γ_3, which is given by (37) and

$$X_0 = \frac{1}{x_1^2}\left\{D\left[\frac{d}{dx_1}(x_2 - \nu_1 x_5), \frac{d}{dx_1}(x_3 - \nu_2 x_5)\right] - \frac{1}{\rho} D[x_2 - \nu_1 x_5, x_3 - \nu_2 x_5, x_4 - \nu_3 x_5]\right\},$$

(39′)
$$X_1 = \frac{1}{x_1^2}\left\{x_1 D\left[\frac{d}{dx_1}(x_2 - \nu_1 x_5), \frac{d}{dx_1}(x_3 - \nu_2 x_5)\right] - \frac{d}{dx_1} D[x_2 - \nu_1 x_5, x_3 - \nu_2 x_5]\right\},$$

$$X_5 = \frac{1}{x_1^2} D[x_5, x_2, x_3].$$

We put

$$D\left[\frac{dx_2}{dx_1}, \frac{dx_3}{dx_1}\right] = x_1^2 \sum_0^\infty I_\nu' x_1^\nu, \qquad D\left[\frac{dx_5}{dx_1}, \frac{dx_3}{dx_1}\right] = x_1^5 \sum_0^\infty J_\nu' x_1^\nu,$$

$$D[x_2, x_3, x_4] = x_1^6 \sum_0^\infty J_\nu x_1^\nu, \qquad \frac{d}{dx_1} D[x_2, x_3] = x_1^3 \sum_0^\infty K_\nu' x_1^\nu,$$

and substitute them in (39′), so that Γ_3 is given by (37) and

$$X_5 = x_1^5 \sum_0^\infty I_\nu x_1^\nu,$$

$$X_1 = x_1 \sum_0^\infty (I_\nu' - K_\nu') x_1^\nu + \nu_2 x_1^3 \sum_0^\infty [E_\nu' - G_\nu'(\nu + 6)] x_1^\nu$$

(39)
$$+ \nu_1 x_1^4 \sum_0^\infty (L_\nu' - J_\nu') x_1^\nu,$$

$$X_0 = \sum_0^\infty I_\nu' x_1^\nu + \nu_2 x_1^2 \sum_0^\infty E_\nu' x_1^\nu - \nu_1 x_1^3 \sum_0^\infty J_\nu' x_1^\nu - \frac{x_1^4}{\rho} \sum_0^\infty J_\nu x_1^\nu + \frac{\nu_3}{\rho} x_1^5 \sum_0^\infty I_\nu x_1^\nu - \frac{\nu_2}{\rho} x_1^6 \sum_0^\infty H_\nu x_1^\nu + \frac{\nu_1}{\rho} x_1^7 \sum_0^\infty F_\nu x_1^\nu.$$

The curve Γ_3 has a singularity S_1^5 at P, whose covariant point O_6 also coincides with P_1. But the covariant points $O_{7,i}$ $(i=1, 2)$ [6] will coincide when and only when

$$\frac{a_0^5}{b_0} U_{10,2} - A_{5,2} = \left(\frac{I_0' - K_0'}{I_0'}\right)^5 \left(\frac{I_2}{I_0} + 4\frac{I_2' + E_0'\nu_2}{I_0'}\right) - 5\left(\frac{I_0' - K_0'}{I_0'}\right)^4 \left(\frac{I_2' - K_2' + \nu_2 E_0' - 6\nu_2 G_0'}{I_0'}\right) = 0. \tag{40}$$

In other words, the plane (37) should lie in a particular four-dimensional space

$$X_2 - \nu_2 X_5 = 0,$$

where ν_2 is the solution of the above linear equation. To select P_5 in this space makes ν_2 vanish, that is

$$\frac{I_0' - K_0'}{I_0'}\left(\frac{I_2}{I_0} + 4\frac{I_2'}{I_0'}\right) - 5\frac{I_2' - K_2'}{I_0'} = 0. \tag{41'}$$

Since

$$\begin{gathered} I_0' = 6b_{2,2}b_{3,3}, \qquad K_0' = 4b_{2,2}b_{3,3}, \\ I_2 = 0, \qquad I_2' = 18b_{2,2}b_{3,5}, \qquad K_2' = 12b_{2,2}b_{3,5}, \end{gathered} \tag{42}$$

(41′) is equivalent to

$$b_{3,5} = 0. \tag{41}$$

If we further impose the condition that the covariant points $O_{8,j}$ $(j=1, 2, 3)$ [6] also coincide

$$\frac{a_0^5}{b_0} U_{10,3} - A_{5,3} = \left(\frac{I_0' - K_0'}{I_0'}\right)^5 \left(\frac{I_3}{I_0} + 4\frac{I_3' - \nu J_0'}{I_0'}\right) - 5\left(\frac{I_0' - K_0'}{I_0'}\right)^4 \left(\frac{I_3' - K_3' + \nu_1 L_0' - \nu_1 J_0'}{I_0}\right) = 0. \tag{43}$$

The singular point P of Γ_3 is then representable only when (40) and (43) are consistent. If we select P_5 in the common part of $x_2=0$ and $x_3-\nu_2 x_5=0$, then ν_1 vanishes,

$$(I_0' - K_0')\left(\frac{I_3}{I_0} + 4\frac{I_3'}{I_0'}\right) - 5(I_3' - K_3') = 0. \tag{44}$$

From (8), (20), (22), (30), (33) and (40) we find that

$$\begin{gathered} I_3' = -30b_{2,5}b_{3,3} + 48b_{2,2}b_{3,6}, \qquad I_3 = 30b_{2,2}b_{3,3}b_{5,5}\left(\frac{b_{5,8}}{b_{5,5}} - \frac{2}{5}\frac{b_{3,6}}{b_{3,3}}\right), \\ K_3' = 28b_{2,2}b_{3,6} - 14b_{2,5}b_{3,3}. \end{gathered} \tag{45}$$

(44) is now simplified to the form

$$b_{5,8}/b_{5,5} + 4b_{2,5}/b_{2,2} - 4b_{3,6}/b_{3,3} = 0. \tag{46}$$

In virtue of (36), (36′) and (46) it can also be written as

$$b_{2,5}/b_{2,2} = 11\omega_1. \tag{47}$$

The coordinates of the osculant O_{10} are

$$\begin{gathered} X_2 = 0, \qquad X_3 = 0, \qquad X_4 = \nu_3, \qquad X_5 = 1, \\ X_1 = -\frac{I_0'}{5I_0}\left[\frac{1}{3}\left(\frac{I_4}{I_0} + 4\frac{I_4'}{I_0'} - \frac{4J_0}{\rho I_0'}\right) - 5\frac{I_4' - K_4'}{I_0'}\right], \\ X_0 = \frac{I_0'}{4I_0}\left[\frac{I_5}{J_0} + 4\left(\frac{I_5'}{I_0'} + \frac{\nu_3 I_0}{\rho I_0'} - \frac{J_1}{\rho I_0'}\right) - 15\frac{I_5' - K_5'}{I_0'}\right], \end{gathered} \tag{48}$$

whose locus is a quadric Q_5:

$$\begin{gathered} X_2=0, \qquad X_3=0, \\ \frac{X_0}{X_5}=\frac{I_0'}{4I_0}\left\{\frac{I_5}{I_0}+\frac{4I_5'}{I_0'}-15\,\frac{I_5'-K_5'}{I_0'}\right\} \\ +\frac{5}{4}\left(\frac{I_0X_4}{J_0X_5}-J_1\right)\left\{\frac{X_1}{X_5}+\frac{I_0'}{5I_0}\left[\frac{1}{3}\left(\frac{I_4}{I_0}+4\,\frac{I_4'}{I_0'}\right)-4\,\frac{I_4'-K_4'}{I_0'}\right]\right\}. \end{gathered} \tag{49}$$

Since P_1 and P_2 lie on the quadric Q_5, there is a point $T(\neq P)$, through which the two generators of Q_5 pass through P_1 and P_4 respectively. We select T as P_5. The equation of the quadric (49) now reduces to

$$X_2 = 0, \qquad X_3 = 0, \qquad X_1X_4 = \frac{4}{45}\cdot\frac{b_{4,4}}{b_{5,5}}X_0X_5. \tag{50}$$

Thus we have

$$\begin{gathered} J_1 = 0, \qquad \frac{1}{3}\left[\frac{I_4}{I_0} + 4\frac{I_4'}{I_0'}\right] - 5\frac{I_4' - K_4'}{I_0'} = 0, \\ \frac{I_5}{I_0} + \frac{4I_5'}{I_0'} - 15\frac{I_5' - K_5'}{I_0'} = 0. \end{gathered} \tag{51}$$

Calculating J_1, I_r, I_r', K_r' $(r=4, 5)$ as well as I_3, I_3' and K_3', we obtain

$$\begin{aligned} &J_1 = 6b_{2,2}b_{3,3}b_{4,5}, \qquad I_4 = 6b_{2,2}b_{3,3}b_{5,5}\left(\frac{4}{3}\,\frac{b_{5,9}}{b_{5,5}} + \frac{b_{2,6}}{b_{2,2}} - 5\,\frac{b_{3,7}}{b_{3,3}}\right), \\ &I_4' = -54b_{2,6}b_{3,3} + 70b_{2,2}b_{3,7}, \qquad K_4' = -24b_{3,3}b_{2,6} + 40b_{2,2}b_{3,7}, \\ &I_5 = 56b_{2,2}b_{3,3}b_{5,10} + 16b_{5,5}b_{2,7}b_{3,5} - 54b_{2,2}b_{5,5}b_{3,8}, \\ &I_5' = -84b_{3,3}b_{2,7} + 98b_{2,2}b_{3,8}, \qquad K_5' = 54b_{2,2}b_{3,8} - 36b_{3,3}b_{2,7}. \end{aligned} \tag{52}$$

Consequently, (51) becomes

$$b_{4,5} = 0, \tag{53}$$

$$7b_{5,9}/b_{5,5} - 100b_{3,7}/3b_{3,3} + 40b_{2,6}/b_{2,2} = 0 \tag{54}$$

and

$$28b_{5,10}/3b_{5,5} + 200b_{2,7}/3b_{2,2} - 50b_{3,8}/b_{3,3} = 0. \tag{55}$$

The pyramid $\{P, P_1, P_2, P_3, P_4, P_5\}$ thus obtained is called the fundamental pyramid of the curve at the point P.

4. **Canonical expansions and projective Frenet-Serret formulae.** There are five species of ordinary curves Γ in a projective five-dimensional space. Every point of Γ may be either an ordinary point of the associated curve C or a k-ic ($k=6, 7, 8$) point. The pyramid in the last section is only valid where every point of Γ is an ordinary point of C, because in other cases the determination of P_2 requires a particular modification.

4.1. *The case where every point of* Γ *is also an ordinary point of the curve* C. Usually, the osculating conic of Γ has at P with C a contact of order equal to or less than four, that is

$$(2D_0B_0/C_0)C_3 - B_0D_3 - B_3D_0 \neq 0. \tag{56}$$

Through a calculation of Wronskians we reach

$$B_3 = 60\prod_3^5 b_{i,i}\left[4\frac{b_{3,6}}{b_{3,3}} - 28\frac{b_{4,7}}{b_{4,4}} + 32\frac{b_{5,8}}{b_{5,5}}\right],$$

$$C_3 = \prod_3^5 b_{i,i}\left[120\frac{b_{3,6}}{b_{3,3}} - 768\frac{b_{4,7}}{b_{4,4}} + 840\frac{b_{5,8}}{b_{5,5}}\right], \tag{57}$$

$$D_3 = \prod_2^5 b_{i,i}\left[48\frac{b_{3,6}}{b_{3,3}} - 240\frac{b_{4,7}}{b_{4,4}} + 240\frac{b_{5,8}}{b_{5,5}}\right],$$

and

$$B_4 = \prod_3^5 b_{i,i}\left[840\frac{b_{3,7}}{b_{3,3}} - 3600\frac{b_{4,8}}{b_{4,4}} + 3240\frac{b_{5,9}}{b_{5,5}}\right],$$

$$C_4 = \prod_3^5 b_{i,i}\left[432\frac{b_{3,7}}{b_{3,3}} - 1680\frac{b_{4,8}}{b_{4,4}} + 1440\frac{b_{5,9}}{b_{5,5}}\right], \tag{58}$$

$$D_4 = \prod_2^5 b_{i,i}\left[-12\frac{b_{2,6}}{b_{2,2}} + 180\frac{b_{3,7}}{b_{3,3}} - 540\frac{b_{4,8}}{b_{4,4}} + 420\frac{b_{5,9}}{b_{5,5}}\right].$$

Substituting (36) and (36′) in (57) we have

$$B_3 = 960\cdot 83\prod_3^5 b_{i,i}\omega_1, \qquad C_3 = 256\cdot 129\prod_3^5 b_{i,i}\omega_1, \qquad D_3 = 33\cdot 256\prod_3^5 b_{i,i}\omega_1.$$

Hence $\omega_1 \neq 0$, because of (56).

We have selected the fundamental pyramid in §3, for which PP_2 is the projective normal of C, so that, as $C_1=0$, $D_2=0$,

$$2(D_0B_0/C_0)C_4 - D_0B_4 - B_0D_4 = 0. \tag{59}$$

From (7) and (59) we obtain

$$-4b_{3,7}/b_{3,3} + 5b_{4,8}/b_{4,4} - 2b_{5,9}/b_{5,5} + b_{2,6}/b_{2,2} = 0. \tag{60}$$

Putting

$$b_{2,6}/b_{2,2} = 244\omega_2 \tag{61}$$

and using the relations (35), (54) and (60), we find

$$b_{3,7}/b_{3,3} = 591\omega_2, \qquad b_{4,8}/b_{4,4} = 992\omega_2, \qquad b_{5,9}/b_{5,5} = 1420\omega_2. \tag{62}$$

The expansions of the nonhomogeneous coordinates of Γ must be

$$\begin{aligned} x_2 &= b_{2,2}[x_1^{2} + 11\omega_1 x_1^{5} + 244\omega_2 x_1^{6}] + b_{2,7}x_1^{7} + (8),\\ x_3 &= b_{3,3}[x_1^{3} + 36\omega_1 x_1^{6} + 591\omega_2 x_1^{7}] + b_{3,8}x_1^{8} + (9),\\ x_4 &= b_{4,4}[x_1^{4} + 72\omega_1 x_1^{7} + 992\omega_2 x_1^{8}] + b_{4,9}x_1^{9} + (10),\\ x_5 &= b_{5,5}[x_1^{5} + 100\omega_1 x_1^{8} + 1420\omega_2 x_1^{9}] + b_{5,10}x_1^{10} + (11). \end{aligned} \tag{63}$$

The unit point in the plane $\{PP_1P_2\}$ can be chosen in the same way as in §1.3. Therefore the cubic (10) is to be given by the equation $-X_0^2X_2+X_1^2X_0/2+X_1X_2^2/2=0$. Hence $D_0B_0=C_0^2/2$, $-2B_0^2C_3/C_0^2D_0+B_0^2D_3/C_0D_0^2+B_0B_3/C_0D_0=1/2$, whence,

$$b_{2,2} = 4/5, \qquad \omega_1 = -1/500. \tag{64}$$

We further select the unit point in space such that (1, 1, 1, 1, 0, 0), (1, 1, 0, 1, 1, 0), (1, 1, 0, 0, 1, 1) lie on Q_3, Q_4 and Q_5 respectively. This implies

$$b_{3,3}/b_{2,2} = 6/5, \qquad b_{4,4}/b_{3,3} = 3/10, \qquad b_{5,5}/b_{4,4} = 4/45. \tag{65}$$

The expansions (63) now are of the form

$$\begin{aligned} x_2 &= \frac{4}{5}\left(x_1^{2} - \frac{11}{4\cdot 5^3}x_1^{5} + 244\omega_2 x_1^{6}\right) + b_{2,7}x_1^{7} + (8),\\ x_3 &= \frac{24}{25}\left(x_1^{3} - \frac{36}{4\cdot 5^3}x_1^{6} + 591\omega_2 x_1^{7}\right) + b_{3,8}x_1^{8} + (9),\\ x_4 &= \frac{36}{125}\left(x_1^{4} - \frac{72}{4\cdot 5^3}x_1^{7} + 992\omega_2 x_1^{8}\right) + b_{4,9}x_1^{9} + (10),\\ x_5 &= \left(\frac{2}{5}\right)^4\left(x_1^{5} - \frac{1}{5}x_1^{8} + 1420\omega_2 x_1^{9}\right) + b_{5,10}x_1^{10} + (11). \end{aligned} \tag{66}$$

Let (P_l) $(l=1, \cdots, 5)$ be the locus of P_l, and let $z_l^j(u)$ $(j=1, \cdots, 6)$ and $z^j(u)$ be the homogeneous coordinates of the curves (P_l) $(l=1, \cdots, 5)$ and Γ with respect to a certain coordinate system; then from the definition of the fundamental pyramid there exist functions a, b and a_{lm} such that

$$(67') \qquad \frac{dz}{du} = az + bz_1, \quad \frac{dz_s}{du} = \sum_{j=0}^{s+1} a_{s,j}z_j, \quad \frac{dz_5}{du} = \sum_{j=0}^{5} a_{5,j}z_j \qquad (s = 1, 2, 3, 4).$$

Evidently there can be determined, save for a constant factor, a unique function $\sigma(u)$ such that

$$d(\sigma z)/du = az_1.$$

Hereafter the coordinates $z^j(u)$ and $z_l^j(u)$ $(l=1, \cdots, 5; j=1, \cdots, 6)$ multiplied by $\sigma(u)$ are still denoted by $z^j(u)$, $z_l^j(u)$. The equations (67′) now reduce to

$$(67) \qquad \frac{dz}{du} = az_1, \quad \frac{dz_s}{du} = \sum_{j=0}^{s+1} a_{s,j}z_j, \quad \frac{dz_5}{du} = \sum_{j=0}^{5} a_{5,j}z_j \qquad (s = 1, 2, 3, 4).$$

If $x_1, \cdots, x_5$ are the nonhomogeneous coordinates of a fixed point in the space with respect to the local coordinate system thus determined, the condition of immovability is

$$\frac{d}{du}\left(z + \sum_{j=1}^{5} x_j z_j\right) = \lambda(u)\left(z + \sum_{j=1}^{5} x_j z_j\right),$$

or

$$(68) \qquad \begin{aligned} \lambda(u) &= \sum_{i=1}^{5} a_{i,0}x_i, \qquad \lambda x_1 = a + \frac{dx_1}{du} + \sum_{i=1}^{5} a_{i,1}x_i, \\ \lambda x_k &= \frac{dx_k}{du} + \sum_{i=1}^{5} a_{i,k}x_i \qquad (k = 2, 3, 4, 5). \end{aligned}$$

Therefore the elimination of λ, dx_1/du and x_k $(k=2, \cdots, 5)$ from (66) and (68) gives four equations. Consequently, we have the relations:

$$(69_1) \qquad \left\{ \begin{aligned} & a_{3,4} = 6a/5, \quad a_{4,4} = 4a_{1,1}, \quad 3a_{1,0} - 16a_{2,1}/5 + 4a_{5,4}/45 = 0, \\ & a_{2,0} - 8a_{3,1}/5 + 21a/50 - a_{3,4}/10 = 0, \\ & 72a_{3,0}/25 - 8\cdot 992 a\omega_2 + 126a_{1,1}/625 - 144a_{4,1}/625 \\ & \qquad + 1970a_{3,4}\omega_2 - 2a_{4,4}/5 = 0, \\ & 36\cdot 548a_{2,1}/5^7 - 18\cdot(2/5)^7 a_{5,1} + 3^5\cdot 2^4 a_{4,0}/5^6 - 3^5\cdot 2^4 a_{1,0}/5^6 \\ & \qquad - 2^{10}\cdot 9\cdot 31 a_{1,1}\omega_2/5^3 - 2^4 a_{5,4}/5^5 - 9ab_{4,9} \\ & \qquad + (2^7\cdot 279/5^3)d\omega_2/du + a_{3,4}b_{3,8} + 279\cdot 2^7 a_{4,4}\omega_2/625 = 0; \end{aligned} \right.$$

$$(69_2)\quad \begin{cases} a_{4,5} = 4a/9, \quad a_{5,5} = 5a_{1,1}, \quad a_{1,0} = a_{2,1}, \\ 2^4a_{2,0} - 24a_{3,1} + 8a - 81a_{4,5}/10 = 0, \\ 96a_{3,0}/25 - 36a_{4,1}/25 + 8a_{1,1}/5 - 11160a\omega_2 + 248\cdot 45a_{4,5}\omega_2 - a_{5,5}/5 = 0, \\ 4^4\cdot 9a_{4,0}/5^7 - 7\cdot 2^4a_{1,0}/5^5 + 11\cdot 2^4a_{2,1}/5^7 - 2^8a_{5,1}/5^7 + 2^9a_{2,1}/5^6 \\ \quad - 10ab_{5,10} + (71\cdot 2^6/5^3)d\omega_2/du + a_{4,5}b_{4,9} - 2^6\cdot 284a_{1,1}\omega_2/5^3 = 0; \end{cases}$$

$$(69_3)\quad \begin{cases} a_{1,2} = 8a/5, \quad a_{2,2} = 2a_{1,1}, \quad a_{1,0} = 2a_{3,2}, \\ 80a_{2,0} - 192a_{3,1} + 11a + 36a_{4,2} = 0, \\ 15\cdot 2^5a_{3,0} - 9\cdot 2^5a_{4,1} + 55a_{1,1} - 732000a\omega_2 - 11a_{2,2} + 16a_{5,2} = 0, \\ 144a_{4,0}/5^4 + a_{1,0}(-88/5^4) + 28\cdot 11a_{2,1}/5^5 - 2^7a_{5,1}/5^5 \\ \quad - 183\cdot 2^5a_{1,1}\omega_2/5 - 7b_{2,7}a + (976/5)d\omega_2/du \\ \quad + 976a_{2,2}\omega_2/5 - 6^3a_{3,2}/5^5 = 0; \end{cases}$$

$$(69_4)\quad \begin{cases} a_{2,3} = 18a/5, \quad a_{3,3} = 3a_{1,1}, \quad 20a_{1,0} - 24a_{2,1} + 3a_{4,3} = 0, \\ 2^6\cdot 15a_{2,0} - 3^3\cdot 2^6a_{3,1} + 6^4a/5 - 11a_{2,3} + 16a_{5,3} = 0, \\ 1152a_{3,0} - 3^3\cdot 2^6a_{3,1}/5 + 6^4a/5 - 11a_{2,3} + 16a_{5,3} = 0, \\ 3^3\cdot 2^6a_{4,0}/5^5 - 6^3a_{1,0}/5^4 + (216\cdot 197/25)d\omega_2/du - 8ab_{3,8} \\ \quad - 197\cdot 288a_{1,1}\omega_2/25 + a_{2,1}(1656/5^6) - 1152a_{5,1}/5^6 \\ \quad + a_{2,3}b_{2,7} - 3^4\cdot 8a_{4,3}/5^6 = 0. \end{cases}$$

Some simple calculations show that (69_i) $(i=1, 2, 3, 4)$ are equivalent to

$$(69')\quad \begin{array}{llll} a_{1,2}=8a/5, & a_{2,3}=18a/5, & a_{3,4}=6a/5, & a_{4,5}=4a/9, \\ a_{2,2}=2a_{1,1}, & a_{3,3}=3a_{1,1}, & a_{4,4}=4a_{1,1}, & a_{5,5}=5a_{1,1}, \\ a_{2,1}=a_{1,0}, & a_{3,2}=a_{1,0}/2, & a_{4,3}=4a_{1,0}/3, & a_{5,4}=9a_{1,0}/4, \\ a_{2,0}=a/10, & a_{3,1}=a/4, & a_{5,3}=291a/40, & a_{4,2}=29a/36, \end{array}$$
$$a_{1,1}=5^3\cdot 3^3a\omega_2/-2,\ a_{3,0}=5^4\cdot 51a\omega_2/8,\ a_{4,1}=2587\cdot 5^3a\omega_2/72,\ a_{5,2}=5^3\cdot 2699a\omega_2/2^5,$$

and

$$(69'')\quad \begin{aligned} 7ab_{2,7} &= 144a_{4,0}/5^4 - 4a_{1,0}/5^4 - 784a_{1,1}\omega_2 + (2^4\cdot 61/5)d\omega_2/du - 2^7a_{5,1}/5^5, \\ 8ab_{3,8} &= 17\cdot 3^3\cdot 2^5a_{4,0}/7\cdot 5^5 + 88\cdot 27a_{1,0}/5^6\cdot 7 + (3^3\cdot 8\cdot 541/27\cdot 5)d\omega_2/du \\ &\quad - 232\cdot 288a_{1,1}\omega_2/25 - 3^4\cdot 2^7a_{5,1}/5^6\cdot 7, \\ 9ab_{4,9} &= -9\cdot 863a_{1,0}/14\cdot 5^7 + 3^4\cdot 472a_{4,0}/5^6\cdot 7 - 2^5\cdot 747a_{5,1}/7\cdot 5^7 \\ &\quad - 603\cdot 2^6a_{1,1}\omega_2/25 + (18\cdot 18757/5^3\cdot 7)d\omega_2/du, \\ 10ab_{5,10} &= 2^5\cdot 799a_{4,0}/5^7\cdot 7 - 5758a_{1,0}/5^7\cdot 63 - 2^7\cdot 209a_{5,1}/5^7\cdot 63 \\ &\quad - 2^6\cdot 3299a_{1,1}\omega_2/125 + (8\cdot 54541/5^3\cdot 63)d\omega_2/du. \end{aligned}$$

By use of (55)

$$
(70)\quad \begin{aligned} a_{4,0} = {} & 50978a_{1,0}/997\cdot 252 - 9256a_{5,1}/6993 + 5^4\cdot 160544a_{1,1}\omega_2/997 \\ & + (5^4\cdot 1334107/36\cdot 997)d\omega_2/du. \end{aligned}
$$

Thus (67) becomes

$$
(\mathrm{A}')\quad \begin{aligned} dz/du &= az_1, \\ dz_1/du &= a(I_1z - 5^3\cdot 3^3I_2z_1/2 + 8z_2/5), \\ dz_2/du &= a(z/10 + I_1z_1 - 5^3\cdot 3^3I_2z_2 + 18z_3/5), \\ dz_3/du &= a(I_2(5^4\cdot 51z/8) + z_1/4 + I_1z_2/2 - 5^3\cdot 3^4I_2z_3/2 + 6z_4/5), \\ dz_4/du &= a\big[(1/997)(50987I_1/252 - 9256I_3/9 - 5^7\cdot 3^3\cdot 80272I_2^2 \\ &\quad + (5^4\cdot 1334107/36)dI_2/adu)z + 2587\cdot 5^3I_2z_1/72 + 29z_2/36 \\ &\quad + 4I_1z_3/3 - 250\cdot 3^3I_2z_4 + 4z_5/9\big], \\ dz_5/du &= a\big[I_4z + I_3z_1 + 5^3\cdot 2099I_2z_2/2^5 + 291z_3/40 \\ &\quad + 9I_1z_4/4 - 5^4\cdot 3^3I_2z_5/2\big], \end{aligned}
$$

where we have denoted $a_{1,0}$, $a_{5,1}$, $a_{5,0}$ and ω_2 by aI_1, aI_3, aI_4 and I_2 respectively. Evidently, a, I_i $(i=1, 2, 3, 4)$ are five projective invariants while adu is an intrinsic form. Putting $adu=d\sigma$ we have

$$
(\mathrm{A})\quad \begin{aligned} dz/d\sigma &= z_1, \\ dz_1/d\sigma &= I_1z - 5^3\cdot 3^3I_2z_1/2 + 8z_2/5, \\ dz_2/d\sigma &= z/10 + I_1z_1 + (-15)^3I_2z_2 + 18z_3/5, \\ dz_3/d\sigma &= 5^4\cdot 51I_2z/8 + z_1/4 + I_1z_2/2 - 5^3\cdot 3^4I_2z_3/2 + 6z_4/5, \\ dz_4/d\sigma &= (1/997)(50987I_1/252 - 9256I_3/9 + (5^4\cdot 1334107/36)dI_2/d\sigma \\ &\quad - 5^7\cdot 3^3\cdot 80272I_2^2)z + 5^3\cdot 2587I_2z_1/72 + 29z_2/36 \\ &\quad + 4I_1z_3/3 - 3^3\cdot 250I_2z_4 + 4z_5/9, \\ dz_5/d\sigma &= I_4z + I_3z_1 + 5^3\cdot 2099I_2z_2/2^5 + 291z_3/40 \\ &\quad + 9I_1z_4/4 - 5^4\cdot 3^3I_2z_5/2. \end{aligned}
$$

Either $(\epsilon_1^2, \epsilon_1, 1)$ or $(\epsilon_2^2, \epsilon_2, 1)$ can be taken for the unit point in the plane $\{PP_1P_2\}$ (cf. §§1.3, 1.4); similarly, each of the two points $(\epsilon_j^2, \epsilon_j, 1, \epsilon_j^2, \epsilon_j, 1)$, $(\epsilon_j^3=1, \epsilon_j\neq 1, j=1, 2)$ can be taken for the unit point in the five-dimensional space. This gives rise to a projective transformation T, under which z, z_1, z_2, z_3, z_4, z_5 correspond to ϵ_j^2z, ϵ_jz_1, z_2, $\epsilon_j^2z_3$, ϵ_jz_4, z_5 respectively. Accordingly $d\sigma$, I_1, I_2 and I_3 are transformed to $\epsilon_jd\sigma$, ϵ_jI_1, $\epsilon_j^2I_2$, ϵ_jI_3 but I_4 is invariant.

Furthermore, elimination of z_1, z_2, z_3, z_4 and z_5 from the equations (A) gives a differential equation of order six (B) invariant with respect to T. The curve in consideration now possesses four invariants I_i $(i=1, \cdots, 4)$ and an arc-element $d\sigma$. Given four analytic functions $I_j(\sigma)$ $(j=1, \cdots, 4)$, the coordinates of a curve in five-dimensional space having $I_j(\sigma)$ as its projective invariants and $d\sigma$ as its projective arc-element are independent solutions of the differential equation (B).

4.2. *The Frenet-Serret formulae in the case where C always has P for its sextactic point.* Since no projective normal can be drawn at a sextactic point of a plane curve, the vertex P_2 in the present case cannot be determined as before. But the way of determining the canonical expansion of a plane curve at its sextactic point furnishes a covariant triangle $\{PP_1P_2\}$ and a unit point in the osculating plane so that the other vertices of the covariant pyramid and the covariant unit point in the five-dimensional space can also be defined without any difficulty.

In fact, from (7) and $2(D_0B_0/C_0)C_3-B_0D_3-B_3D_0=0$, we have

$$\omega_1 = 0. \tag{71}$$

Hence neither (59) nor (60) exists.

From the definition of the triangle $\{PP_1P_2\}$ and (21) and (23) in §2.1 it follows that the quartic

$$\begin{aligned} 8X_2^3(X_2 - 2X_1) - 8X_2^3(2X_1 - 2X_0 - X_2/2) \\ + (2X_1 - 2X_0 - X_2/2)^2(4X_1^2 - 4X_0X_2) = 0 \end{aligned} \tag{72}$$

has at P a contact of order seven with C. Since $B_i=0$, $C_i=0$, $D_i=0$ $(i=1, 2, 3)$, we obtain after substituting (4) in (72) that

$$\begin{aligned} C_0^2 - B_0D_6 = 0, \qquad D_0^3 + B_0(2C_0C_4 - B_0D_4 - B_4D_0) = 0, \\ 2C_0C_5 - B_0D_5 - B_5D_0 = 0, \end{aligned} \tag{73}$$

which are equivalent to

$$b_{2,2} = 8/5, \tag{74}$$

$$(2/5)^4 + b_{2,6}/b_{2,2} - 4b_{3,7}/b_{3,3} + 5b_{4,8}/b_{4,4} - 2b_{5,9}/b_{5,5} = 0, \tag{75}$$

$$8b_{2,7}/b_{2,2} - 20b_{3,8}/b_{3,3} + 20b_{4,9}/b_{4,4} - 7b_{5,10}/b_{5,5} = 0, \tag{76}$$

because of (7) and

$$B_5 = 60 \prod_3^5 b_{i,i}\left[2^5\cdot\frac{b_{3,8}}{b_{3,3}} - 3^3\cdot 4\frac{b_{4,9}}{b_{4,4}} + 84\frac{b_{5,10}}{b_{5,5}}\right],$$

$$C_5 = \prod_3^5 b_{i,i}\left[7\cdot 144\,\frac{b_{3,8}}{b_{3,3}} - 6\cdot 8^3\frac{b_{4,9}}{b_{4,4}} + 28\cdot 3^4\frac{b_{5,10}}{b_{5,5}}\right],$$

$$D_5 = 48 \prod_2^5 b_{i,i}\left[-\frac{b_{2,7}}{b_{2,2}} + 9\frac{b_{3,8}}{b_{3,3}} - 21\frac{b_{4,9}}{b_{4,4}} + 14\frac{b_{5,10}}{b_{5,5}}\right].$$

Combining (65) and (74) we obtain

$$b_{3,3} = 48/25, \qquad b_{4,4} = 72/125, \qquad b_{5,5} = 2^5/5^4. \tag{77}$$

For convenience we put

$$b_{5,9}/b_{5,5} = \omega_2; \tag{78}$$

then (35), (54), (75) and (77) give

$$\begin{aligned} b_{2,6}/b_{2,2} &= 61\omega_2/355 + (45/71)\cdot(2/5)^4, \\ b_{3,7}/b_{3,3} &= 591\omega_2/1420 + (54/71)\cdot(2/5)^4, \\ b_{4,8}/b_{4,4} &= 248\omega_2/355 + (20/71)\cdot(2/5)^4. \end{aligned} \tag{79}$$

The expansions of the coordinates of Γ are no longer(66), but take the form

$$\begin{aligned} x_2 &= \frac{8}{5}\left[\overset{2}{x_1} + \left(\frac{61\omega_2}{355} + \frac{45}{71}\cdot\left(\frac{2}{5}\right)^4\right)\overset{6}{x_1}\right] + b_{2,7}\overset{7}{x_1} + (8), \\ x_3 &= \frac{48}{25}\left[\overset{3}{x_1} + \left(\frac{591\omega_2}{1420} + \frac{54}{71}\cdot\left(\frac{2}{5}\right)^4\right)\overset{7}{x_1}\right] + b_{3,8}\overset{8}{x_1} + (9), \\ x_4 &= \frac{72}{125}\left[\overset{4}{x_1} + \left(\frac{248\omega_2}{355} + \frac{20}{71}\cdot\left(\frac{2}{5}\right)^4\right)\overset{8}{x_1}\right] + b_{4,9}\overset{9}{x_1} + (10), \\ x_5 &= \frac{2^5}{5^4}\left[\overset{5}{x_1} + (\omega_2)\overset{9}{x_1}\right] + b_{5,10}\overset{10}{x_1} + (11). \end{aligned} \tag{80}$$

However, there are, as in the former case, functions $a_{ij}(u)$ and $a(u)$ satisfying (67). Some computations suffice to demonstrate that

$$\begin{array}{llll} a_{1,2} = 16a/5, & a_{2,3} = 18a/5, & a_{3,4} = 6a/5, & a_{4,5} = 4a/9, \\ a_{2,2} = 2a_{1,1}, & a_{3,3} = 3a_{1,1}, & a_{4,4} = 4a_{1,1}, & a_{5,5} = 5a_{1,1}, \\ a_{2,1} = a_{1,0}/2, & a_{3,2} = a_{1,0}/2, & a_{4,3} = 4a_{1,0}/3, & a_{5,4} = 9a_{1,0}/4, \\ a_{2,0} = 0, & a_{3,1} = 0, & a_{4,2} = 0, & a_{5,3} = 0, \end{array} \tag{81}$$

as well as eight other relations

$$\begin{aligned} &-\frac{72}{25}a_{4,1} - 3a\left(\frac{61\omega_2}{71} + \frac{144}{71\cdot 25}\right) + \frac{2}{25}\cdot a_{5,2} = 0, \\ &-\frac{72}{25}a_{4,1} + \frac{32}{5}a_{3,0} - \frac{1135a}{284}\omega_2 - \frac{6^4}{71\cdot 5^3}a = 0, \\ &-\frac{72}{25}a_{4,1} + \frac{36}{5}a_{3,0} - \frac{1393a}{284}\omega_2 + \frac{2^5\cdot 7}{71\cdot 5^3}a = 0, \\ &-\frac{24}{25}a_{4,1} + \frac{64}{25}a_{3,0} - \frac{391a}{213}\omega_2 + \frac{2^6}{213\cdot 25}a = 0, \end{aligned} \tag{81'}$$

$$\frac{b_{2,7}}{b_{2,2}} = \frac{1}{7a}\left(a_{4,0}\cdot\frac{72}{125} - \frac{2^6}{5^4}a_{5,1} - 4a_{1,1}\frac{b_{2,6}}{b_{2,2}}\right),$$

$$\frac{b_{3,8}}{b_{3,3}} = \frac{1}{8a}\left[\frac{72\cdot 31}{5^3\cdot 7}a_{4,0} - \frac{3^3\cdot 2^5}{7\cdot 5^4}a_{5,1} + 4a_{1,1}\left(-\frac{b_{3,7}}{b_{3,3}} - \frac{3}{7}\frac{b_{2,6}}{b_{2,2}}\right)\right],$$

$$(81'') \quad \frac{b_{4,9}}{b_{4,4}} = \frac{1}{9a}\left[\frac{73\cdot 36}{5^3\cdot 7}a_{4,0} - \frac{2^4\cdot 83}{5^4\cdot 7}a_{5,1} + 2a_{1,1}\left(-\frac{2b_{4,8}}{b_{4,4}} - \frac{b_{3,7}}{b_{3,3}} - \frac{3}{7}\frac{b_{2,6}}{b_{2,2}}\right)\right],$$

$$\frac{b_{5,10}}{b_{5,5}} = \frac{1}{10a}\left[\frac{-2^5\cdot 30187}{9^2\cdot 5^4\cdot 7}a_{4,0} - \frac{2^5\cdot 128239}{5^5\cdot 7\cdot 3^6}a_{5,1}\right.$$
$$\left. + \frac{2^5 a_{1,1}}{5\cdot 9^3}\left(-\frac{9^3\cdot 5}{8}\frac{b_{5,9}}{b_{5,5}} - \frac{2b_{4,8}}{b_{4,4}} - \frac{b_{3,7}}{b_{3,3}} - \frac{3}{7}\frac{b_{2,6}}{b_{2,2}}\right)\right].$$

From (81′) we find

$$(82) \quad \begin{aligned} \omega_2 &= 2^6\cdot 5413/525\cdot 4421, \\ a_{3,0} &= (6a/5)\cdot[2^4\cdot 19/71\cdot 5^2 - 129\omega_2/142], \\ a_{4,1} &= a(1358/3195 - 25\cdot 3199\omega_2/71\cdot 9\cdot 2^5), \\ a_{5,2} &= a(2^4\cdot 407/355 + 61675\omega_2/568). \end{aligned}$$

Substituting (81″) in (55) and (76) we are led to

$$(83) \quad a_{4,0} = C_1 a_{1,1}, \qquad a_{5,1} = C_2 a_{1,1},$$

where C_1 and C_2 are two constants. If the invariant form adu be denoted by $d\sigma$, and the projective invariants $a_{1,0}/a$, $a_{1,1}/a$ and $a_{5,0}/a$ by I_1, I_2 and I_3 respectively, then the projective Frenet-Serret formulae read:

$$\frac{dz}{d\sigma} = z_1, \qquad \frac{dz_1}{d\sigma} = I_1 z + I_2 z_1 + \frac{16}{5}z_2, \qquad \frac{dz_2}{d\sigma} = \frac{1}{2}I_1 z_1 + 2I_2 z_2 + \frac{18}{5}z_3,$$

$$(\mathrm{C}) \quad \begin{aligned} \frac{dz_3}{d\sigma} &= \frac{6}{355}\left(\frac{16\cdot 19}{25} - \frac{129}{2}\omega_2\right)z + \frac{I_1}{2}z_2 + 3I_2 z_3 + \frac{6}{5}z_4, \\ \frac{dz_4}{d\sigma} &= C_1 I_2 z + \frac{1}{639}\left(\frac{1358}{5} - \frac{5^2\cdot 3199}{2^5}\omega_2\right)z_1 + \frac{4I_1}{3}z_3 + 4I_2 z_4 + \frac{4}{9}z_5, \end{aligned}$$

$$\frac{dz_5}{d\sigma} = I_3 z + C_2 I_2 z_1 + \frac{25}{71}\left(-\frac{2467}{8}\omega_2 + \frac{2^4\cdot 407}{5^3}\right)z_2 + \frac{9I_1}{4}z_4 + 5I_2 z_5.$$

In this case four points can be taken for the unit point without changing the invariants $I_1(\sigma)$, $I_2(\sigma)$, $I_3(\sigma)$ and the form $d\sigma$ save for a constant factor. The differential equation obtained from (C) by eliminating z_i $(i=1, 2, 3, 4, 5)$ is evidently independent of these various selections(5).

(5) We can reach similar results in §§4.3 and 4.4.

4.3. *The case where every point of* Γ *is a 7-ic point of the curve* C. When P is a 7-ic point, the osculating conic of Γ at P has at P a contact of order six with C. Besides $\omega_1=0$, we have

$$(84)\qquad 2(B_0D_0/C_0)C_4 - B_0D_4 - B_4D_0 = 0$$

or its equivalent

$$(85)\qquad -4b_{3,7}/b_{3,3} + 5b_{4,8}/b_{4,4} + b_{2,6}/b_{2,2} - 2b_{5,9}/b_{5,5} = 0.$$

We now have to take the fundamental triangle of the plane curve C at P as the triangle $\{PP_1P_2\}$ and select the unit point in this plane by the method stated in §2.2. This implies that the quartic $-2X_1X_2^3/3+X_0^2(X_1^2-X_0X_2)=0$ has at P a contact of order eight with C. We then have the following conditions

$$(86)\qquad \begin{gathered} C_0^2 - B_0D_0 = 0, \qquad -2C_0D_0^3/3 + B_0^2(2C_0C_5 - B_0D_5 - B_5D_0) = 0, \\ 2C_0C_6 - B_0D_6 - B_6D_0 = 0, \end{gathered}$$

which, in turn, can be expressed in terms of $b_{i,j}$ as follows:

$$b_{2,2} = 8/5,$$

$$(87)\qquad -\frac{2^4}{3\cdot 5^5} + 8\frac{b_{2,7}}{b_{2,2}} - 20\frac{b_{3,8}}{b_{3,3}} + 20\frac{b_{4,9}}{b_{4,4}} - 7\frac{b_{5,10}}{b_{5,5}} = 0,$$

$$(88)\qquad 12\frac{b_{2,8}}{b_{2,2}} + 5\frac{b_{3,9}}{b_{3,3}} - 21\frac{b_{4,10}}{b_{4,4}} + 28\cdot\left(\frac{3}{5}\right)^2\frac{b_{5,11}}{b_{5,5}} = 0,$$

use being made of (7) and

$$(89)\qquad \begin{aligned}
B_5 &= \prod_3^5 b_{i,i}\left[15\cdot 2^7\frac{b_{3,8}}{b_{3,3}} - 80\cdot 3^4\frac{b_{4,9}}{b_{4,4}} + 5040\frac{b_{5,10}}{b_{5,5}}\right],\\
C_5 &= \prod_3^5 b_{i,i}\left[1008\frac{b_{3,8}}{b_{3,3}} - 3\cdot 2^{10}\frac{b_{4,9}}{b_{4,4}} + 28\cdot 3^4\frac{b_{5,10}}{b_{5,5}}\right],\\
D_5 &= \prod_2^5 b_{i,i}\left[-3\cdot 2^4\frac{b_{2,7}}{b_{2,2}} + 3^3\cdot 16\frac{b_{3,8}}{b_{3,3}} - 63\cdot 16\frac{b_{4,9}}{b_{4,4}} + 21\cdot 2^5\frac{b_{5,10}}{b_{5,5}}\right],\\
B_6 &= \prod_3^5 b_{i,i}\left[3600\frac{b_{3,9}}{b_{3,3}} - 10500\frac{b_{4,10}}{b_{4,4}} + 231\cdot 2^5\frac{b_{5,11}}{b_{5,5}}\right],\\
C_6 &= \prod_3^5 b_{i,i}\left[1920\frac{b_{3,9}}{b_{3,3}} - 5040\frac{b_{4,10}}{b_{4,4}} + 210\cdot 16\frac{b_{5,11}}{b_{5,5}}\right],\\
D_6 &= \prod_2^5 b_{i,i}\left[-120\frac{b_{2,8}}{b_{2,2}} + 840\frac{b_{3,9}}{b_{3,3}} - 1680\frac{b_{4,9}}{b_{4,4}} + 63\cdot 16\frac{b_{5,11}}{b_{5,5}}\right].
\end{aligned}$$

Since $b_{2,2}$ has the same numerical value as in the former case, the equations

(77) and (81) remain valid. From (35), (54) and (85) we find, by putting

$$b_{5,9}/b_{5,5} = \omega_2,$$

that

(90) $b_{2,6}/b_{2,2} = 61\omega_2/71\cdot 5, \quad b_{3,7}/b_{3,3} = 591\omega_2/1420, \quad b_{4,8}/b_{4,4} = 248\omega_2/355.$

(81′) must now be replaced by the four independent equations

$$\begin{aligned} - 72a_{4,1}/25 - 183a\omega_2/71 + 2a_{5,2}/25 &= 0, \\ - 72a_{4,1}/25 + 2^5 a_{3,0}/5 - 1135a\omega_2/284 &= 0, \\ - 72a_{4,1}/25 + 36a_{3,0}/5 - 1393a\omega_2/284 &= 0, \\ - 24a_{4,1}/25 + 64a_{3,0}/25 - 391a\omega_2/213 &= 0, \end{aligned} \tag{91}$$

linear in ω_2, $a_{4,1}$, $a_{5,2}$ and $a_{3,0}$. Thus

(92) $\omega_2 = 0, \qquad a_{4,1} = 0, \qquad a_{5,2} = 0, \qquad a_{3,0} = 0.$

Consequently (81″) reduce to

$$\begin{aligned} a_{4,0}\cdot\frac{72}{125} - \frac{64}{625}\,a_{5,1} - \frac{7ab_{2,7}}{b_{2,2}} &= 0, \\ a_{4,0}\cdot\frac{144}{125} - \frac{96}{625}\,a_{5,1} + 3a\frac{b_{2,7}}{b_{2,2}} - 8a\,\frac{b_{3,8}}{b_{3,3}} &= 0, \\ a_{4,0}\cdot\frac{216}{125} - \frac{2^7}{5^4}\,a_{5,1} + 4a\frac{b_{3,8}}{b_{3,3}} - 9a\,\frac{b_{4,9}}{b_{4,4}} &= 0, \\ a_{4,0}\cdot\frac{288}{125} - \frac{2^5}{125}\,a_{5,1} + 5a\frac{b_{4,9}}{b_{4,4}} - 10a\,\frac{b_{5,10}}{b_{5,5}} &= 0. \end{aligned} \tag{93}$$

Eliminating $b_{2,7}/b_{2,2}$, $b_{3,8}/b_{3,3}$, $b_{4,9}/b_{4,4}$, $b_{5,10}/b_{5,5}$ from (55), (87) and (93) we obtain

(94) $3^3\cdot 8189a_{4,0} - 508a_{5,1} = 0, \qquad a_{5,1} - 14a/5 - 28179a_{4,0}/2 = 0,$

whence

(95) $a_{4,0} = D_1 a, \qquad a_{5,1} = D_2 a,$

where D_1 and D_2 are constants.

In summary, we have simplified (63) and (67) as

$$\begin{aligned} x_2 &= 8x_1^2/5 + b_{2,7}x_1^7 + b_{2,8}x_1^8 + (9), \\ x_3 &= 48x_1^3/25 + b_{3,8}x_1^8 + b_{3,9}x_1^9 + (10), \\ x_4 &= 72x_1^4/125 + b_{4,9}x_1^9 + b_{4,10}x_1^{10} + (11), \\ x_5 &= 2^5x_1^5/5^4 + b_{5,10}x_1^{10} + b_{5,11}x_1^{11} + (12) \end{aligned} \tag{63'}$$

and

$$
(67') \quad \begin{aligned}
&\frac{dz}{du} = az_1, \qquad \frac{dz_1}{du} = a\left(I_1 z + I_2 z_1 + \frac{16}{5} z_2\right),\\
&\frac{dz_2}{du} = a\left(\frac{1}{2} I_1 z_1 + 2I_2 z_2 + \frac{18}{5} z_3\right), \quad \frac{dz_3}{du} = a\left(\frac{1}{2} I_1 z_2 + 3I_2 z_3 + \frac{6}{5} z_4\right),\\
&\frac{dz_4}{du} = a\left(D_1 z + \frac{4}{3} I_1 z_3 + 4I_2 z_4 + \frac{4}{9} z_5\right),\\
&\frac{dz_5}{du} = a\left(I_3 z + D_2 z_1 + \frac{9}{4} I_1 z_4 + 5I_2 z_5\right).
\end{aligned}
$$

From the condition of immovability we obtain that

$$
(96) \quad \begin{aligned}
2^8 I_3/5^5 - 5b_{2,7} - 8b_{2,8} &= 0,\\
6\cdot 2^9 I_3/5^6 - 5I_2 b_{3,8} + 18b_{2,8}/5 - 9b_{3,9} &= 0,\\
3^3\cdot 2^8 I_3/5^7 - 5I_2 b_{4,9} + 6b_{3,9}/5 - 10b_{4,10} &= 0,\\
4b_{4,10}/9 - 5I_2 b_{5,10} - 11b_{5,11} &= 0.
\end{aligned}
$$

Eliminating $b_{i,i+j}$ $(i=2, 3, 4, 5; j=5, 6)$ from (89) and (96) we obtain

$$
-135I_2(1557a_{4,0} - 172a_{5,1}) + I_3(36\cdot 511 + 55\cdot 973/2) = 0.
$$

Hence follow the projective Frenet-Serret formulae:

$$
(D) \quad \begin{aligned}
&\frac{dz}{d\sigma} = z_1, \qquad \frac{dz_1}{d\sigma} = \left(I_1 z + I_2 z_1 + \frac{16}{5} z_2\right),\\
&\frac{dz_2}{d\sigma} = \left(\frac{1}{2} I_1 z_1 + 2I_2 z_2 + \frac{18}{5} z_3\right), \quad \frac{dz_3}{d\sigma} = \left(\frac{1}{2} I_1 z_2 + 3I_2 z_3 + \frac{6}{5} z_4\right),\\
&\frac{dz_4}{d\sigma} = \left(D_1 z + \frac{4}{3} I_1 z_3 + 4I_2 z_4 + \frac{4}{9} z_5\right),\\
&\frac{dz_5}{d\sigma} = \left(D_3 I_2 z + D_2 z_1 + \frac{9}{4} I_1 z_4 + 5I_2 z_5\right),
\end{aligned}
$$

where D_1, D_2, D_3 are constants.

In this case five points can be taken for the unit point. I_1, I_2 and $d\sigma$ are uniquely determined save for a constant factor.

4.4. *The case where every point of* Γ *is an* 8-*ic point of the curve* C. The necessary and sufficient condition that the analytic point P of C is an 8-ic point are $\omega_1=0$, (85) and

$$
(97) \qquad 2(B_0 D_0/C_0)C_5 - B_0 D_5 - B_5 D_0 = 0.
$$

Now the covariant vertex P_2 can be determined only by the projective differential theory of 8-ic points. In case that the triangle of reference $\{PP_1P_2\}$

and the unit point $P+P_1+P_2$ are selected as in §2.2, the quartic

$$X_2^4 + 4X_0^2(X_1^2 - X_0X_2) = 0 \tag{98}$$

must have at P a contact of order nine with C. Consequently

$$\begin{gathered} C_0^2 - B_0D_0 = 0, \qquad D_0^4 + 4B_0^2(2C_0C_6 - B_0D_6 - B_6D_0) = 0, \\ 2C_0C_7 - B_0D_7 - B_7D_0 = 0. \end{gathered} \tag{99}$$

In virtue of (7), (89) and

$$B_7 = \prod_3^5 b_{i,i}\left[6000\,\frac{b_{3,10}}{b_{3,3}} - 2^4\cdot 990\,\frac{b_{4,11}}{b_{4,4}} + 2^7\cdot 3^4\,\frac{b_{5,12}}{b_{5,5}}\right],$$

$$C_7 = \prod_3^5 b_{i,i}\left[3240\,\frac{b_{3,10}}{b_{3,3}} - 7680\,\frac{b_{4,11}}{b_{4,4}} + 176\cdot 3^3\,\frac{b_{5,12}}{b_{5,5}}\right],$$

$$D_7 = \prod_2^5 b_{i,i}\left[-480\,\frac{b_{2,9}}{b_{2,2}} + 1440\,\frac{b_{3,10}}{b_{3,3}} - 81\cdot 32\,\frac{b_{4,11}}{b_{4,4}} + 1440\,\frac{b_{5,12}}{b_{5,5}}\right],$$

the equations (99) reduce to

$$b_{2,2} = 8/5, \tag{100}$$

$$8/5^6 + 25b_{3,9}/b_{3,3} - 105b_{4,10}/b_{4,4} + (252/5)b_{5,11}/b_{5,5} + 60b_{2,8}/b_{2,2} = 0 \tag{101}$$

and

$$-155b_{3,10}/b_{3,3} + 80b_{2,9}/b_{2,2} + 244b_{4,11}/b_{4,4} + (126/5)b_{5,12}/b_{5,5} = 0 \tag{102}$$

respectively. (77), (81) and (92) still remain valid, but the second equation of (94) should be replaced by

$$2a_{5,1} - 28179a_{4,0} = 0,$$

because of replacing (87) by (97). Hence

$$a_{5,1} = 0, \qquad a_{4,0} = 0; \tag{103}$$

and a priori,

$$b_{2,7} = 0, \qquad b_{3,8} = 0, \qquad b_{4,9} = 0, \qquad b_{5,10} = 0.$$

The equations (96) then reduce to

$$\begin{gathered} 2^5I_3 - 5^5b_{2,8} = 0, \qquad 2^{10}I_3 + 6\cdot 5^5b_{2,8} - 3\cdot 5^6b_{3,9} = 0, \\ 3^3\cdot 2^7I_3 + 3\cdot 5^6b_{3,9} - 5^8b_{4,10} = 0, \qquad 4b_{4,10}/9 - 11b_{5,11} = 0. \end{gathered} \tag{104}$$

Eliminating $b_{2,8}$, $b_{3,9}$, $b_{4,10}$ and $b_{5,11}$ from (101) and (104) we find

$$I_3 = 198a5\cdot 529\cdot 673. \tag{105}$$

The conditions of immovability imply

$$(106)\qquad \begin{aligned} &2I_2b_{2,8} + 3b_{2,9} = 0, \qquad 3I_2b_{3,9} + 5b_{3,10} - 9b_{2,9}/5 = 0,\\ &6I_2b_{4,10} + 11b_{4,11} - 6b_{3,10}/5 = 0, \qquad 6I_2b_{5,11} + 12b_{5,12} - 4b_{4,11}/9 = 0, \end{aligned}$$

which cannot be consistent with (102) unless

$$(107)\qquad I_2 = 0.$$

In summary, we obtain the canonical expansions of Γ:

$$(63'')\qquad \begin{aligned} x_2 &= 8x_1^2/5 + b_{2,8}x_1^8 + b_{2,10}x_1^{10} + (11),\\ x_3 &= 48x_1^3/25 + b_{3,9}x_1^9 + b_{3,11}x_1^{11} + (12),\\ x_4 &= 72x_1^4/125 + b_{4,10}x_1^{10} + b_{4,12}x_1^{12} + (13),\\ x_5 &= 2^5x_1^5/5^4 + b_{5,11}x_1^{11} + b_{5,13}x_1^{13} + (14), \end{aligned}$$

where $b_{2,8}$, $b_{3,9}$, $b_{4,10}$ and $b_{5,11}$ are constants given by (104) and (105), and the projective Frenet-Serret formulae:

$$(\mathrm{E})\qquad \begin{aligned} &\frac{dz}{d\sigma} = z_1, \qquad \frac{dz_1}{d\sigma} = I_1 z + \frac{16}{5}z_2, \qquad \frac{dz_2}{d\sigma} = \frac{I_1}{2}z_1 + \frac{18}{5}z_3,\\ &\frac{dz_3}{d\sigma} = \frac{1}{2}I_1z_2 + \frac{6}{5}z_4, \qquad \frac{dz_4}{d\sigma} = \frac{4}{3}I_1z_3 + \frac{4}{9}z_5,\\ &\frac{dz_5}{d\sigma} = \frac{198}{5\cdot 529\cdot 673}z + \frac{9}{4}I_1z_5. \end{aligned}$$

4.5. *The case where every point of* Γ *is a 9-ic point of the curve C.* We are now in a position to deal with the remaining case. In the same manner as before PP_1 is taken as the tangent to Γ at P, and P_1P_2 as the tangent to the osculating conic at one of its points P_2, and $P+P_1+P_2$ on the osculating conic.

When every point P of Γ is a 9-ic point of C, the equations $\omega_1=0$, (85), (97) and

$$(108)\qquad 2C_0C_6 - B_0D_6 - B_6D_0 = 0$$

must be consistent. After multiplying the coordinates of the points (z), (z_1), $\cdots$, (z_5) with suitable factor $\rho(u)$, the equations (75) remain valid. Because (77), (81), (92) and (103) are satisfied here, (63) and (67) become

$$(109)\qquad \begin{aligned} &x_2 = \frac{8}{5}x_1^2 + \sum_{l=8}^{\infty} b_{2,2+l}x_1^{2+l}, \qquad x_3 = \frac{48}{25}x_1^3 + \sum_{l=8}^{\infty} b_{3,3+l}x_1^{3+l},\\ &x_4 = \frac{72}{125}x_1^4 + \sum_{l=8}^{\infty} b_{4,4+l}x_1^{4+l}, \qquad x_5 = \frac{2^5}{5^4}x_1^5 + \sum_{l=8}^{\infty} b_{5,5+l}x_1^{5+l}, \end{aligned}$$

and

$$\frac{dz}{du} = az_1, \qquad \frac{dz_1}{du} = a\left(I_1 z + I_2 z_1 + \frac{16}{5} z_2\right),$$

$$(110) \qquad \frac{dz_2}{du} = a\left(\frac{I_1}{2} z + 2I_2 z_2 + \frac{18}{5} z_3\right), \qquad \frac{dz_3}{du} = a\left(\frac{I_1}{2} z_1 + 3I_2 z_3 + \frac{6}{5} z_4\right),$$

$$\frac{dz_4}{du} = a\left(\frac{4}{3} I_1 z_3 + 4I_2 z_4 + \frac{4}{9} z_5\right), \qquad \frac{dz_5}{du} = a\left(\frac{9I_1}{4} z_4 + 5I_2 z_5\right).$$

We have to treat particularly the condition of immovability

$$\frac{d}{du}\left(z + \sum_{j=1}^{5} x_j z_j\right) = \lambda(u)\left(z + \sum_{j=1}^{5} x_j z_j\right).$$

Substituting (110) in it,

$$\lambda = aI_1 x_1, \qquad \lambda x_1 = a + \frac{dx_1}{du} + aI_2 x_1 + \frac{a}{2} I_1 x_2,$$

$$\lambda x_2 = \frac{16}{5} a x_1 + \frac{dx_2}{du} + 2aI_2 x_2 + \frac{aI_1}{2} x_3,$$

$$(111)$$

$$\lambda x_3 = \frac{18}{5} a x_2 + \frac{dx_3}{du} + 3aI_2 x_3 + \frac{aI_1}{3} x_4,$$

$$\lambda x_4 = \frac{6}{5} a x_3 + \frac{dx_4}{du} + 4aI_2 x_4 + \frac{aI_1}{4} x_5, \quad \lambda x_5 = \frac{4}{9} a x_4 + \frac{dx_5}{du} + 5aI_2 x_5,$$

and eliminating λ and x_2 from the first two equations of (111), we obtain

$$(112) \qquad \frac{dx_1}{du} = -a - aI_2 x_1 + \frac{9}{5} aI_1 x_1^2 + \frac{aI_1}{2} \sum_{l=8}^{\infty} b_{2,2+l} x_1^{2+l}.$$

Therefore the third of (111) is found to be equivalent to

$$(113) \qquad \begin{aligned} I_1 a x_1\left(\frac{8}{5} x_1^2 + \sum_{l=8}^{\infty} b_{2,2+l} x_1^{2+l}\right) &= \frac{16}{5} a x_1 + 2aI_2\left(\frac{8}{5} x_1^2 + \sum_{l=8}^{\infty} b_{2,2+l} x_1^{2+l}\right) \\ &+ \left[\frac{16}{5} x_1 + \sum_{l=8}^{\infty} (2 + l) b_{2,2+l} x_1^{1+l}\right] \\ &\cdot\left[-a - aI_2 x_1 + \frac{9a}{5} I_1 x_1^2 + \frac{aI_1}{2} \sum_{l=8}^{\infty} b_{2,2+l} x_1^{2+l}\right] \\ &+ \frac{aI_1}{2}\left[\frac{48}{25} x_1^3 + \sum_{8}^{\infty} b_{3,3+l} x_1^{3+l}\right] \end{aligned}$$

on account of $\lambda = a x_1 I_1$, (112) and the first two equations of (109). Comparing in (113) the coefficients of equal powers in x_1, we see that I_1 and $b_{2,2+l}$ ($l=8, 9, \cdots$) vanish. From the last three equations of (111), we find similarly that

$$b_{3,3+l}=0, \qquad b_{4,4+l}=0, \qquad b_{5,5+l}=0 \qquad (l=8,9,\cdots).$$

Finally, the expansions of Γ become

$$x_2 = 8x_1^2/5, \qquad x_3 = 48x_1^3/25, \qquad x_4 = 72x_1^4/125, \qquad x_5 = 2^5x_1^5/5^2.$$

We may easily find the geometrical construction of them by extending the method used for a space curve in ordinary space.

4.6. *The geometrical interpretation of the invariants and the arc-element.* The unit point geometrically defined in the foregoing sections is available in the interpretation of the invariants and the arc-element. In fact, projection from P_2 of the consecutive point of P on Γ to the line PP_1 produces $P+d\sigma P_1$ save for infinitesimal of higher order. Evidently

$$(P, P_1; P + d\sigma P_1, P + P_1) = d\sigma.$$

This furnishes a geometrical interpretation of the projective arc-element.

As to the interpretation of the invariants we utilize the last equation of the Frenet-Serret formulae. For example, from the equation

$$\frac{dz_5}{d\sigma} = I_4 z + I_3 z_1 + \frac{5^3\cdot 2699}{2^5} I_2 z_2 + \frac{291}{40} z_3 + \frac{9}{4} I_1 z_4 - \frac{5^4\cdot 3^3}{2} I_2 z_5,$$

it can be shown that the line PP_3 intersects the hyperplane containing P_1, P_2, P_4 and the tangent at P_5 to the curve (P_5) at the point $I_4P+291P_1/40$ and the double ratio $(P, P_3; P+P_3, I_4P+291P_3/40)$ equals, except for a constant factor, the invariant I_4. Since other invariants may be similarly interpreted we omit their interpretations here.

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